Model Structures on Diagram Categories

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For Laura, Athena, Hermies, Artemis, and Hera.

Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

Model categories have been an important tool in algebraic topology since first defined by Quillen. Given a category and a class of morphisms called weak equivalences one can study the homotopy "category" in which the weak equivalences are turned into isomorphisms by formally giving them inverses. However, the resulting structure might not be a category, and even when it is understanding it can be very difficult. A model structure on a category ensures that formally inverting the weak equivalence does result in a category. It also makes the study of the homotopy category easier by providing two weak factorisation systems on the model category which can be used to understand the homotopy category.

We explore the basic consequences of weak factorisation systems and show how one can be cofibrantly generated from a set of morphisms. We then define model categories and discuss some fundamental results about them, including defining their homotopy categories, and proving a recognition theorem. Having done this we show there is a cofibrantly generated model structure on the category of compactly generated, weakly Hausdorff, topological spaces, $\mathcal{T}$.

We take a look at the category of simplicial sets, **sSet**, which can be considered a generalisation of inductively constructed topological spaces. We later describe a cofibrantly generated model structure on them and a Quillen adjunction between $\mathcal{T}$ and **sSet**.

In stable homotopy theory the important objects of study are categories of $\mathcal{D}$-spectra and the stable model structures on them. We define a level model structure on $\mathcal{D}$-spectra for chosen categories $\mathcal{D}$, explain why it is not satisfactory for stable homotopy theory, and then describe the stable model structure on spectra.

Finally, we describe the Reedy model structure on diagram categories $\mathcal{M}^{\mathcal{C}}$

where $\mathcal{M}$ is a model category and $\mathcal{C}$ is a Reedy category. A recent result classifying those functors between Reedy categories which induce a Quillen functor between diagram categories for any choice of model category by Hirschhorn and Volić is shown using a dual argument to the one in their paper.

Contents

Chapter 1

Some Category Theory

We assume some basic knowledge of category theory of the reader.

Definition 1.1. Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. We say that $\mathcal{D}$ is a **full subcategory** of $\mathcal{C}$ if, for all objects X and Y in $\mathcal{D}$, $f \in \mathcal{C}(X, Y) \implies f \in \mathcal{D}(X, Y)$.

Definition 1.2. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. Then the **comma category** of F over G, denoted $(F \downarrow G)$, is the category whose

- objects are triples (α, β, f) such that $\alpha \in \mathrm{obj}(\mathcal{A})$, $\beta \in \mathrm{obj}(\mathcal{B})$, and $f : F(\alpha) \to G(\beta)$ is a morphism in $\mathcal{C}$;
- morphisms $(\alpha, \beta, f) \to (\alpha', \beta', f')$ are pairs (ν, μ) where $\nu : \alpha \to \alpha'$ in $\mathcal{A}$ and $\mu : \beta \to \beta'$ in $\mathcal{B}$ such that the following diagram commutes.

$$\begin{tikzcd} F(\alpha) \arrow[r, "F\nu"] \arrow[d, "f"'] & F(\alpha') \arrow[d, "f'"] \\ G(\beta) \arrow[r, "G\mu"'] & G(\beta') \end{tikzcd}$$

Notation 1.3. We will sometimes refer to objects of comma categories by their morphism component, leaving the object components implied.

Comma categories involving functors out of the category $\mathbf{1}$ with a single object and morphism, subcategory inclusion functors, and identity functors are particularly important.

For a functor $\mathbf{1} \to \mathcal{C}$ picking out a single object $\alpha \in \mathrm{obj}(\mathcal{C})$ by an abuse of notation the chosen object is written in place of the functor in comma category notation. As there is only one object and one morphism in $\mathbf{1}$, objects in a comma category where the source of one of the functors is $\mathbf{1}$ can be treated as pairs and similarly morphisms can be considered a single morphism in the source category of the other functor.

When one of the functors is an identity functor on a category, the category is written in place of the identity functor. If one of the functors is a subcategory inclusion functor the subcategory is written in place of the identity functor. In the case where the image of the other functor is contained in the subcategory this can lead to some unfortunate confusion. For example let $F : \mathbf{1} \to \mathcal{C}$ and $G : \mathcal{B} \hookrightarrow \mathcal{C}$ be the inclusion of $\mathcal{B}$ a subcategory of $\mathcal{C}$ and $F(\mathbf{1}) = \alpha \in \mathrm{obj}(\mathcal{B})$. Then $(\alpha \downarrow \mathcal{B})$ could either be $(\alpha \downarrow 1_{\mathcal{B}})$ or it could be $(\alpha \downarrow G)$.

Notation 1.4. Unless otherwise specified, when a subcategory is indicated it will denote the identity functor on that subcategory and not the inclusion functor.

Definition 1.5. Let $\mathcal{C}$ be a category, and let α be an object of $\mathcal{C}$. The **slice category** of $\mathcal{C}$ over α, $(\mathcal{C} \downarrow \alpha)$, is the comma category where $\mathcal{C}$ indicates the identity functor $1_{\mathcal{C}}$. Similarly **coslice category** of $\mathcal{C}$ under α, $(\alpha \downarrow \mathcal{C})$, is the comma category where $\mathcal{C}$ indicates the identity functor $1_{\mathcal{C}}$.

Definition 1.6. Let $F : \mathcal{C} \to \mathcal{M}$ be a functor. A **cocone** under F is an object X in $\mathcal{M}$ together with a set of morphisms $\{\mu_\alpha : F\alpha \to X\}_{\alpha \in \mathrm{obj}(\mathcal{C})}$ such that for all morphisms $\varphi : \alpha \to \beta$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd} F\alpha \arrow[rr, "F\varphi"] \arrow[dr, "\mu_\alpha"'] & & F\beta \arrow[dl, "\mu_\beta"] \\ & X & \end{tikzcd}$$

In fact this describes a natural transformation $\mu : F \implies \Delta X$ where ΔX is the constant functor $\mathcal{C} \to \mathcal{M}$ taking all objects to X and all morphisms to 1_X. Hence a cocone is an object X together with such a natural transformation. Dually, a **cone** over F is an object X in $\mathcal{M}$ together with a natural transformation $\nu : \Delta X \implies F$.

Definition 1.7. Given a functor $F : \mathcal{C} \to \mathcal{M}$ we can consider the category of cocones under F. A morphism between cocones $(X, \mu) \to (Y, \omega)$ is a morphism $f : X \to Y$ in $\mathcal{M}$ such that for all objects α in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd} & F\alpha \arrow[dl, "\mu_\alpha"'] \arrow[dr, "\omega_\alpha"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

If the category of cocones has an initial object (X, μ) then it is the **colimit** of F and we write $\operatorname{colim} F = X$. Dually, we can consider the category of cones over F. A morphism between cones $(X, \mu) \to (Y, \omega)$ is a morphism $f : X \to Y$ in $\mathcal{M}$ such that for all objects α in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mu_\alpha"'] & & Y \arrow[dl, "\omega_\alpha"] \\ & F\alpha & \end{tikzcd}$$

If the category of cones has a terminal object (X, μ) then it is the **limit** of F and we write $\lim F = X$. As initial and terminal objects colimits and limits are unique up to unique isomorphism.

Where a (co)limit is written with a subcategory as a subscript then it is the (co)limit of the functor restricted to that subcategory. That is, if $\mathcal{B}$ is a subcategory of $\mathcal{C}$ with inclusion functor $\iota_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathcal{C}$ then $\operatorname*{colim}_{\mathcal{B}} F = \operatorname{colim}(F\iota_{\mathcal{B}})$ and $\lim_{\mathcal{B}} F = \lim(F\iota_{\mathcal{B}})$.

Examples 1.8. (i) Where $\mathcal{C}$ is the empty category with no objects. If they exist, the limit and colimit of the unique functor $\mathcal{C} \to \mathcal{M}$ are the terminal and initial objects of $\mathcal{M}$ respectively.

(ii) **Products/coproducts.** If $\mathcal{C} = \{\bullet \quad \bullet\}$, that is the category with two objects and only identity morphisms, a functor $F : \mathcal{C} \to \mathcal{M}$ picks out two objects, A, B in $\mathcal{M}$. Then the limit of F is the product of A and B written $A \times B$ and the colimit of F is the coproduct of F written $A \amalg B$.

(iii) **Pushouts.** If $\mathcal{C} = \{ \begin{smallmatrix} \bullet & \to & \bullet \\ \downarrow & & \\ \bullet & & \end{smallmatrix} \}$ the image of a functor $F : \mathcal{C} \to \mathcal{M}$ is a diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "k"] & B \\ C & \end{tikzcd}$$

in $\mathcal{M}$. The colimit of F, if it exists, is an object $B \amalg_A C$ together with morphisms $\iota_B : B \to B \amalg_A C$ and $\iota_C : C \to B \amalg_A C$ in $\mathcal{M}$ such that for any $X \in \operatorname{obj}(\mathcal{M})$ and morphisms $B \to X$ and $C \to X$ making the outside of the following diagram commute there exists a unique morphism $B \amalg_A C \to X$ such that the whole diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "k"'] & B \arrow[d, "\iota_B"'] \arrow[ddr, bend left] & \\
C \arrow[r, "\iota_C"] \arrow[drr, bend right] & B \amalg_A C \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}$$

We say that ι_B is a pushout of k along j.

(iv) **Pullbacks.** If $\mathcal{C} = \{ \begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet & \rightarrow \bullet \end{smallmatrix} \}$ then the image of a functor $F : \mathcal{C} \to \mathcal{M}$ is a diagram

$$\begin{tikzcd}
& B \arrow[d, "j"] \\
C \arrow[r, "k"] & A
\end{tikzcd}$$

in $\mathcal{M}$. The limit of F, if it exists, is an object $B \times_A C$ together with morphisms $\iota_B : B \times_A C \to B$ and $\iota_C : B \times_A C \to C$ in $\mathcal{M}$ such that for any $X \in \operatorname{obj}(\mathcal{M})$ and morphisms $X \to B$ and $X \to C$ making the outside of the following diagram commute there exists a unique morphism $X \to B \times_A C$ such that the whole diagram commutes.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& B \times_A C \arrow[r, "\iota_B"'] \arrow[d, "\iota_C"] & B \arrow[d, "j"] \\
& C \arrow[r, "k"'] & A
\end{tikzcd}$$

(v) **Equalisers/Coequalisers.** If $\mathcal{C} = \{\bullet \rightrightarrows \bullet\}$ a functor $F : \mathcal{C} \to \mathcal{M}$ picks out two objects, A, B in $\mathcal{M}$ and two morphisms $f, g : A \to B$. The limit of F is the equaliser of f and g. The colimit of F is the coequaliser of f and g. The coequaliser, if it exists, is an object C of $\mathcal{M}$ together with a map $h : B \to C$ satisfying $hf = hg$ such that for any object $D \in \operatorname{obj} \mathcal{M}$ and map $h' : B \to D$ satisfying $h'f = h'g$ there is a unique morphism making the following diagram commute.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[dr, "h'"'] & C \arrow[d, dashed, "\exists!"] \\
& & D
\end{tikzcd}$$

Proposition 1.9. *Let $f : A \to D$, $g : A \to B$, and $h : B \to C$ be morphisms in some category $\mathcal{M}$. Let $i : B \to E$ be the pushout of f along g. Then $j : C \to F$ is the pushout of i along h if and only if j is the pushout of f along hg.*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "h"] \arrow[d, "i"] & C \arrow[d, "j"] \\ D \arrow[r, "k"'] & E \arrow[r, "l"'] & F \end{tikzcd}$$

Proof. Suppose j is the pushout of i along h. Let $\varphi : D \to W$ and $\psi : C \to W$ be morphisms in $\mathcal{M}$ such that $\varphi f = \psi hg$. Then as i is a pushout of f along g the morphisms φ and ψh induce a unique morphism $\mu : E \to W$ such that $\mu k = \varphi$ and $\mu i = \psi h$. As j is a pushout of i along h the morphisms μ and ψ induce a unique morphism $\sigma : F \to W$ such that $\sigma l = \mu$ and $\sigma j = \psi$ so $\varphi = \mu k = \sigma lk$. Note that because μ is uniquely determined by φ and ψ, so is σ. Hence j is a pushout of f along hg.

Conversely, suppose that j is a pushout of hg. Let $\mu : E \to W$ and $\psi : C \to W$ be morphisms in $\mathcal{M}$ such that $\psi h = \mu i$. Then we have $\mu kf = \mu ig = \psi hg$, as j is a pushout of f along hg the morphisms μk and ψ induce a unique morphism $\sigma : F \to W$ such that $\sigma lk = \mu k$ and $\sigma j = \psi$. We have $\mu i = \psi h = \sigma jh = \sigma li$. As i is a pushout of f along g the equalities $\sigma lk = \mu k$ and $\mu i = \sigma li$ mean that $\sigma l = \mu$. Hence j is a pushout of i along h. □

Definition 1.10. We call a category $\mathcal{M}$ **complete** if it contains all small limits, that is if $\lim F$ exists for all functors $F : \mathcal{C} \to \mathcal{M}$ where $\mathcal{C}$ is a small category. Similarly, we call a category **cocomplete** if it contains all small colimits, that is if $\operatorname{colim} F$ exists for all functors $F : \mathcal{C} \to \mathcal{M}$ where $\mathcal{C}$ is a small category. If a category is both complete and cocomplete we call it **bicomplete**.

Definition 1.11. Let $\mathcal{M}$ be a category, and let $\mathcal{C}$ be a small category. Then we have a category whose objects are functors $X : \mathcal{C} \to \mathcal{M}$ and whose morphisms are natural transformations $f : X \to Y$ with composition defined as a vertical composition of natural transformations. We call this the category of **$\mathcal{C}$-diagrams in $\mathcal{M}$**, denoted $\mathcal{M}^{\mathcal{C}}$. Categories of this type are called **diagram categories**.

For any category $\mathcal{M}$ a functor $G : \mathcal{C} \to \mathcal{D}$ between small categories induces a functor $G^* : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ taking functors $X : \mathcal{D} \to \mathcal{M}$ to $XG : \mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{X} \mathcal{M}$.

Definition 1.12. Let X be an object in a category $\mathcal{M}$. An object Y in $\mathcal{M}$ is a **retract** of X if there exist morphisms $i : Y \to X$ and $r : X \to Y$ such that $ri = 1_Y$.

Remark 1.13. Given a morphism $f : X \to Y$ in a category $\mathcal{M}$ a retract of f is a retract in the arrow category $\mathcal{M}^{\mathbf{2}}$. That is, a morphism $g : A \to B$ in $\mathcal{M}$ such that there exists a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "1_A"] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r] \arrow[rr, bend right, "1_B"'] & Y \arrow[r] & B \end{tikzcd}$$

Definition 1.14. Let $\mathcal{M}$ a cocomplete category and λ be some limit ordinal. We can consider λ to be the category whose objects are ordinals $\beta \leq \lambda$ with $\lambda(\alpha, \beta) = \{*\}$ if $\alpha \leq \beta$ and $\lambda(\alpha, \beta) = \varnothing$ otherwise. A **λ-sequence** in $\mathcal{C}$ is a functor $X : \lambda \to \mathcal{C}$ such that for every limit ordinal γ the induced morphism $\operatorname*{colim}_{\beta<\gamma} \to X_\gamma$ is an isomorphism. The **composition** of X is the morphism $X_0 \to \operatorname*{colim}_{\beta<\lambda} X_\beta$.

Definition 1.15. Let $\mathcal{J}$ be a class of morphisms in a cocomplete category $\mathcal{M}$. A **λ-sequence** in $\mathcal{J}$ is a λ-sequence in $\mathcal{M}$ such that the morphism $X_\beta \to X_{\beta+1}$ is in $\mathcal{J}$ for all $\beta < \lambda$. A **transfinite composition** of morphisms in $\mathcal{J}$ is the composition of a λ-sequence in $\mathcal{J}$.

Proposition 1.16. *Let $\mathcal{J}$ be a class of morphisms in a cocomplete category $\mathcal{M}$ and $f : A \to B$ be a morphism in $\mathcal{M}$. If there exists a limit ordinal γ and a functor $X : \gamma \to \mathcal{M}$ such that*

- $X_0 = A$,
- $\operatorname*{colim}_{\beta<\gamma} X_\beta = B$,
- *the natural morphism $X_0 \to \operatorname*{colim}_{\beta<\gamma} X_\beta$ is f,*
- *and for all $\beta + 1 < \gamma$ the induced morphism $\operatorname*{colim}_{\alpha\leq\beta} X_\alpha \to X_{\beta+1}$ is in $\mathcal{J}$,*

then f is a transfinite composition of morphisms in $\mathcal{J}$.

Proof. Let λ be the smallest ordinal such that the morphism $X_\beta \to X_{\beta+1}$ is the identity for $\lambda \leq \beta + 1 < \gamma$. Then restricting X to $\lambda \to X$ gives a λ-sequence in $\mathcal{J}$ whose composition is f. Hence f is a transfinite composition of morphisms in $\mathcal{J}$. $\square$

Definition 1.17. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors, we write $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$. We say that F is a **left adjoint functor** to G (and that G is a **right adjoint functor** to F) if for all objects in X in $\mathcal{C}$ and all objects Y in $\mathcal{D}$ there exists bijections $\varphi_{X,Y} : \mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$ which are natural in X and Y. To be explicit, naturality in X and Y here mean that for any morphisms $f : W \to X$ in $\mathcal{C}$ and $g : Y \to Z$ in $\mathcal{D}$ the following diagrams commute.

$$\begin{array}{ccc} \mathcal{D}(FX,Y) & \xrightarrow{\varphi_{X,Y}} & \mathcal{C}(X,GY) \\ \Big\downarrow{\scriptstyle (Ff)^*} & & \Big\downarrow{\scriptstyle (f)^*} \\ \mathcal{D}(FW,Y) & \xrightarrow{\varphi_{W,Y}} & \mathcal{D}(W,GY) \end{array} \qquad \begin{array}{ccc} \mathcal{D}(FX,Y) & \xrightarrow{\varphi_{X,Y}} & \mathcal{C}(X,GY) \\ \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle (Gg)_*} \\ \mathcal{D}(FX,Z) & \xrightarrow{\varphi_{X,Z}} & \mathcal{D}(X,GZ) \end{array}$$

We have a bifunctor $\mathcal{D}(F-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ which sends $(X, Y) \in \mathrm{obj}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D})$ to $\mathcal{D}(FX, Y)$ and morphisms $(f^{\mathrm{op}} : X \to W, g : Y \to Z)$ to $(FX \xrightarrow{h} Y) \mapsto (FW \xrightarrow{Ff} FX \xrightarrow{h} Y \xrightarrow{g} Z)$. Similarly, we have a bifunctor $\mathcal{C}(-, G-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ which sends $(X, Y) \in \mathrm{obj}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D})$ to $\mathcal{C}(X, GY)$ and morphisms $(f^{\mathrm{op}} : X \to W, g : Y \to Z)$ to $(X \xrightarrow{h} GY) \mapsto (W \xrightarrow{f} X \xrightarrow{h} GY \xrightarrow{Gg} GZ)$. With these bifunctors in mind the bijections above are the components of a natural isomorphism $\varphi : \mathcal{D}(F-,-) \to \mathcal{C}(-, G-)$. We call the triple (F, G, φ) an **adjunction** between $\mathcal{C}$ and $\mathcal{D}$.

Notation 1.18. We write $F \dashv G$ to indicate that F is a left adjoint functor to G.

Definition 1.19. Let (F, G, φ) be an adjunction between $\mathcal{C}$ and $\mathcal{D}$ where $F \dashv G$ and $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$. The **unit** of the adjunction is the natural transformation $\eta : 1_{\mathcal{C}} \to GF$ whose components are $\eta_X = \varphi_{X,F(X)}(1_{F(X)})$. The **counit** of the adjunction is the natural transformation $\varepsilon : FG \to 1_{\mathcal{D}}$ whose components are $\varepsilon_Y = \varphi^{-1}(1_{G(Y)})$.

Definition 1.20. Let $\mathcal{C}$ be a bicomplete category. We define the category $\mathcal{C}_+$ as the comma category $(* \downarrow \mathcal{C})$ where $*$ is the terminal object in $\mathcal{C}$. This category is also bicomplete. There is an obvious functor $\mathcal{C} \to \mathcal{C}_+$ sending an object X to $X_+ = X \coprod *$, that is, it adds a disjoint basepoint. This functor is left adjoint to the forgetful functor $U : \mathcal{C}_+ \to \mathcal{C}$ which takes an object and 'forgets' that it has a basepoint.

Definition 1.21. A **monoidal category** is a sextuple $(\mathcal{M}, \otimes, \mathcal{I}, \lambda, \rho, \alpha)$ where $\mathcal{M}$ is a category, $\otimes$ is a bifunctor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, I is a distinguished object called the unit object, λ is a natural isomorphism with components $\lambda_X : I \otimes X \cong X$ called the left unitor, ρ is a natural isomorphism with components $\rho_X : X \otimes I \cong X$, and α is a natural isomorphism (natural in X, Y, and Z) with components $\alpha_{X,Y,Z} : ((X \otimes Y) \otimes Z) \cong (X \otimes (Y \otimes Z))$ called the associator and satisfying two coherence conditions. The first coherence condition is that for any objects W, X, Y, Z in $\mathcal{M}$ the following diagram commutes.

$$\begin{tikzcd}
& (W \otimes (X \otimes Y)) \otimes Z \arrow[dr, "\alpha_{W,X\otimes Y,Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W,X,Y}\otimes 1_Z"] \arrow[d, "\alpha_{W\otimes X,Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[d, "1_W \otimes \alpha_{X,Y,Z}"] \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[rr, "\alpha_{W,X,Y\otimes Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

The second coherence condition is that for any objects X and Y in $\mathcal{M}$ the following diagram commutes.

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

In general we we refer to a monoidal category $(\mathcal{M}, \otimes, I, \lambda, \rho, \alpha)$ just by the underlying category $\mathcal{M}$.

Definition 1.22. A **symmetric** monoidal category is a monoidal category $\mathcal{M}$ together with an natural isomorphism (natural in X and Y) with components $s_{X,Y} : X \otimes Y \cong Y \otimes X$ called the braiding satisfying the coherence conditions that for any objects X, Y, and Z in $\mathcal{M}$ the following diagrams commute.

$$\begin{tikzcd}
X \otimes I \arrow[rr, "s_{X,I}"] \arrow[dr, "\rho_X"'] & & I \otimes X \arrow[dl, "\lambda_X"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \otimes X \arrow[dr, "s_{Y,X}"] & \\
X \otimes Y \arrow[ur, "s_{X,Y}"] \arrow[rr, equal] & & X \otimes Y
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,X}"] \arrow[d, "s_{X,Y}\otimes 1_Z"'] & X \otimes (Y \otimes Z) \arrow[r, "s_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "\alpha_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[r, "1_Y \otimes s_{X,Z}"'] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

Definition 1.23. A **closed** symmetric monoidal category is a symmetric monoidal category $\mathcal{M}$ such that for all objects X in $\mathcal{M}$ the functor $-\otimes X : \mathcal{M} \to \mathcal{M}$ has a right adjoint, $\hom(X, -) : \mathcal{M} \to \mathcal{M}$ called **internal hom**.

Example 1.24. The category, $\mathcal{T}$ of compactly generated, weakly Hausdorff, based topological spaces is a closed symmetric monoidal category. Here the monoidal product is the smash product $X \wedge Y = (X \times Y)/(X \vee Y)$, the unit object is the zero dimensional sphere S^0, and for $X, Y \in \mathrm{obj}(\mathcal{T})$ the internal hom $\hom(X, Y)$ is $\mathcal{T}(X, Y)$ with the compact-open topology.

Proposition 1.25. *If $\mathcal{C}$ is a small category and $\mathcal{M}$ is a complete category, then for any functor $G : \mathcal{C} \to \mathcal{D}$ the induced functor $G^* : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ has a right adjoint.*

For a proof of this proposition see Corollary 2 on page 235 of [9].

Definition 1.26. A **zig-zag** between two objects X and Y in a category $\mathcal{M}$ is a finite sequence of morphisms.

$$\begin{tikzcd}
& X_1 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & \cdots \arrow[dl, "f_3"] \arrow[dr, "f_{n-1}"'] & & X_n \arrow[dl, "f_n"'] \arrow[dr, "f_{n+1}"] & \\
X & & X_2 & & X_{n-1} & & Y
\end{tikzcd}$$

A category $\mathcal{M}$ is **connected** if given any two objects there exists a zig-zag between them.

Definition 1.27. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, and let $G : \mathcal{C} \to \mathcal{D}$ be a functor. The functor G is **left cofinal** (or **initial**) if for all objects α in $\mathcal{D}$ the comma category $(G \downarrow \alpha)$ is non-empty and connected. The functor G is **right cofinal** (or **terminal**) if for all objects α in $\mathcal{D}$ the comma category $(\alpha \downarrow G)$ is non-empty and connected.

For a proof of the following proposition see [3] Proposition 14.2.5.

Proposition 1.28. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, and let $G : \mathcal{C} \to \mathcal{D}$ be a functor.*

(i) The functor G is left cofinal if and only if for every functor $X : \mathcal{D} \to \mathcal{M}$ where $\mathcal{M}$ is a complete category the natural morphism $\lim_{\mathcal{D}} X \to \lim_{\mathcal{C}} G^*X$ is an isomorphism.

(ii) The functor G is right cofinal if and only if for every functor $X : \mathcal{D} \to \mathcal{M}$ where $\mathcal{M}$ is a cocomplete category the natural morphism $\operatorname{colim}_{\mathcal{C}} G^*X \to \operatorname{colim}_{\mathcal{D}} X$ is an isomorphism.

Chapter 2

Simplicial Sets

Definition 2.1. The **delta category**, Δ, is the category whose objects are the ordered sets $[n] = \{0, 1, \ldots, n\}$ for each $n \in \mathbb{N} = \{0, 1, \ldots\}$. A morphism $[n] \to [m]$ is a weakly order preserving function from $[n]$ to $[m]$.
There are two important sets of morphisms in Δ. The **coface morphisms** $d^{i,n}$ defined for all $n \in \mathbb{N}$ and $0 \leq i \leq n$

$$d^{i,n} : [n-1] \to [n], k \mapsto \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

and the **codegeneracy morphisms** $s^{i,n}$ defined for all $n \in \mathbb{N}$ and $0 \leq i \leq n$

$$s^{i,n} : [n+1] \to [n], k \mapsto \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i. \end{cases}$$

Notation 2.2. In general these morphisms are written d^i and s^i as the codomain is usually clear from context.

For a proof of the following proposition see page 173 of [9].

Proposition 2.3. *The coface and codegeneracy morphisms of Δ satisfy the following relations*

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, \quad \textit{if } i < j \\ s^j s^i &= s^i s^{j+1}, \quad \textit{if } i < j \\ s^j d^i &= \begin{cases} 1, & \textit{if } i = j, j+1 \\ d^i s^{j-1}, & \textit{if } i < j \\ d^{i-1} s^j, & \textit{if } i > j+1. \end{cases} \end{aligned}$$

Further, any morphism $f : [n] \to [m]$ in Δ can be factored uniquely as a composite of coface morphisms followed by a composite of codegeneracy morphisms. This factorisation has the form

$$f = s^{i_1} \dots s^{i_k} d^{j_1} \dots d^{j_l}$$

such that $n - l + k = m$, $0 \leq i_k < \dots < i_1 < m$, and $0 \leq j_1 < \dots < j_l < n - 1$.

Definition 2.4. Let $\mathcal{M}$ be a category. Then a **simplicial object** in $\mathcal{M}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{M}$. We will write $\mathbf{s}\mathcal{M}$ for the category of simplicial objects in $\mathcal{M}$, note that $\mathbf{s}\mathcal{M}$ is the diagram category $\mathcal{M}^{\Delta^{\mathrm{op}}}$. If X is a simplicial object in $\mathcal{M}$ we call the morphisms $d_{i,n} = Xd^{i,n}$ and $s_{i,n} = Xs^{i,n}$ the face morphisms and the degeneracy morphisms respectively.

The most important example of a category of simplicial objects is the category of simplicial sets, **sSet**. We will consider this category in some detail. Let X be a simplicial set, that is a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$. The information of X consists of sets $X_n = X[n]$ for all $n \in \mathbb{N}$ and, by Proposition 2.3, for all $n \in \mathbb{N}$ and $0 \leq i \leq n$ face and degeneracy morphisms $d_{i,n}$ and $s_{i,n}$ satisfying the following relations (as in Δ we omit n where it is clear from context)

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i, \quad \text{if } i < j \\
s_i s_j &= s_{j+1} s_i, \quad \text{if } i < j \\
d_i s_j &= \begin{cases} 1, & \text{if } i = j, j+1 \\ s_{j-1} d_i, & \text{if } i < j \\ s_j d_{i-1}, & \text{if } i > j+1. \end{cases}
\end{aligned}$$

We will refer to these as the **simplicial relations**. Let X, Y be simplicial sets, then a morphism $f : X \to Y$ in **sSet** is a natural transformation with components $f_n : X_n \to Y_n$. In order for f to be a natural transformation of functors $X \implies Y$ we must have that the diagram

$$\begin{tikzcd}
X_n \arrow[r, "Xg"] \arrow[d, "f_n"'] & X_m \arrow[d, "f_m"] \\
Y_n \arrow[r, "Yg"'] & Y_m
\end{tikzcd}$$

commutes for all morphisms $g : [m] \to [n]$ in Δ. However by Proposition 2.3 g factors as a composite of coface and codegeneracy morphisms. So the above diagram factors into squares as in the following diagram.

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "f_n"'] & \dots \arrow[r] & X_p \arrow[r, "d_i"] \arrow[d, "f_p"] & X_{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & \dots \arrow[r] & X_r \arrow[r, "s_j"] \arrow[d, "f_r"] & X_{r+1} \arrow[r] \arrow[d, "f_{r+1}"] & \dots \arrow[r] & X_m \arrow[d, "f_m"] \\
Y_n \arrow[r] & \dots \arrow[r] & Y_p \arrow[r, "d_i"] & Y_{p-1} \arrow[r] & \dots \arrow[r] & Y_r \arrow[r, "s_j"] & Y_{r+1} \arrow[r] & \dots \arrow[r] & Y_m
\end{tikzcd}$$

Hence it is sufficient that the squares

$$\begin{tikzcd}
X_n \arrow[r, "d_i"] \arrow[d, "f_n"'] & X_{n-1} \arrow[d, "f_{n-1}"] \\
Y_n \arrow[r, "d_i"'] & Y_{n-1}
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
X_n \arrow[r, "s_i"] \arrow[d, "f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\
Y_n \arrow[r, "s_i"'] & Y_{n+1}
\end{tikzcd}$$

commute for all $n \in \mathbb{N}$ and all $0 \leq i \leq n$.

If X is a simplicial set we say that the elements of X_n are the **n-simplices** of X. This terminology is for $\mathbf{s}\mathcal{M}$ where the objects of the underlying category $\mathcal{M}$ are sets with structure on them. We say that an n-simplex is degenerate if it is in the image of some degeneracy morphism, otherwise we say it is non-degenerate.

In **sSet** the empty simplicial set $\varnothing : \Delta^{\mathrm{op}} \to \mathbf{Set}$ with $\varnothing_n = \varnothing$ for all $n \in \mathbb{N}$ is an initial object and the simplicial set $* : \Delta^{\mathrm{op}} \to \mathbf{Set}$ with $*_n = \{*\}$ for all $n \in \mathbb{N}$ is a terminal object. In the case of $\varnothing \in \mathrm{obj}(\mathbf{sSet})$ it is sufficient to specify that $\varnothing_0$ is the empty set because there is only one function $\varnothing \to \varnothing$ in **Set**. For $* \in \mathrm{obj}(\mathbf{sSet})$ it is sufficient to say that there is a single non-degenerate 0-simplex and that all other simplices are degenerate, the simplicial relations ensure there is a single element in $*_n$ for all $n \in \mathbb{N}$. We show this by induction. If $a, b \in *_1$ then, as they must be degenerate simplices, they are in the image of a degeneracy morphism. There is only one degeneracy morphism $*_0 \to *_1$ and $*_0$ is a singleton, so $a = s_0(*) = b$. Hence $*_1$ is a singleton. Now let $n \geq 2$ and suppose that $*_k$ is a singleton for all $k < n$. Let $a, b \in *_n$, as all n-simplices are degenerate $a = s_i(*)$ and $b = s_j(*)$ for some $0 \leq i, j \leq n$. Relabelling if necessary, let $i \leq j$. Clearly if $i = j$ then $a = s_i(*) = s_j(*) = b$. Otherwise $i < j$ and by the simplicial relations and the fact that $*_{n-1} = *_{n-2} = \{*\}$ we have $a = s_i(*) = s_i s_{j-1}(*) = s_j s_i(*) = s_j(*) = b$.

In fact, by the simplicial relations, a simplicial set is entirely determined by its non-degenerate simplices and their images under the relevant face morphisms.

The representable functors $\Delta^n = \Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathbf{Set}, [k] \mapsto \Delta([k], [n])$ are an important set of simplicial sets. The non-degenerate simplices are the injective morphisms $[k] \to [n]$ in Δ, so Δ^n has $\binom{n+1}{k+1}$ non-degenerate k-simplices. The face morphisms $d_i : \Delta([k], [n]) \to \Delta([k-1], [n])$ are given by precomposition by the coface morphisms so that $d_i([k] \xrightarrow{f} [n]) = ([k-1] \xrightarrow{fd^i} [n])$. Similarly the degeneracy morphisms $s_i : \Delta([k], [n]) \to \Delta([k+1], [n])$ are given by precomposition by the codegeneracy morphisms so that $s_i([k] \xrightarrow{f} [n]) = ([k+1] \xrightarrow{fs^i} [n])$. The **boundary** of Δ^n, written $\partial\Delta^n$, is the simplicial set Δ^n with the single non-degenerate n-simplex removed. That is,

$$\partial\Delta^n_k = \Delta([k], [n]) = \begin{cases} \Delta([k], [n]) & \text{if } k < n \\ \Delta([n], [n]) \backslash \{1_{[n]}\} & \text{if } k = n \\ \Delta([k], [n]) \backslash S_k & \text{if } k > n \end{cases}$$

where S_k is the set of surjective morphisms $[k] \to [n]$. The S_k sets consist precisely of those degenerate simplices in Δ^n generated by the n-simplex $1_{[n]}$. The non-degenerate $(n-1)$-simplices are precisely the coface morphisms $d^i : [n-1] \to [n]$. For $0 \leq 1 \leq n$ the i**th horn** of Δ^n, written Λ^n_i, is the boundary $\partial\Delta^n$ with the $(n-1)$-simplex d^i removed.

Definition 2.5. For each $n \in \mathbb{N}$ there is an n**th evaluation functor** ev_n : **sSet** $\to$ **Set** with $ev_n(X) = X_n$ and $ev_n(f) = f_n$. In the other direction there is an n**th simplicial set functor** $(-_n)_\bullet$: **Set** $\to$ **sSet** where $(X_n)_\bullet$ is the simplicial set whose only non-degenerate simplices are the elements of X which are n-simplices and a single non-degenerate 0-simplex. Note that there is a single simplex in $(X_n)_{n-1}$ so the face morphisms are the same and take all n-simplices to the $n-1$ simplex. Given a morphism $f : X \to Y$ of sets, $(f_n)_\bullet : (X_n)_\bullet \to (Y_n)_\bullet$ has components $(f_n)_i : \{*\} \to *$ for $0 \leq i < n$ and $(f_n)_n = f : X \to Y$. The remaining components are determined by the simplicial relations.

Proposition 2.6. *The nth evaluation functor and the the nth simplicial set functor are adjoint functors $ev_n \dashv (-_n)_\bullet$.*

Proof. Let $X \in \mathrm{obj}(\mathbf{sSet})$ and $Y \in \mathrm{obj}(\mathbf{Set})$. Suppose we have a morphism $f : X \to (Y_n)_\bullet$ in **sSet**. Below degree n the components of f are morphisms in **Set** into a singleton set and so are uniquely determined. Above degree n the components of f are entirely determined by the component of f at degree n as $(Y_n)_i$ contains only degenerate simplices for $i > n$. Hence f is determined by its nth component which is a morphism $f_n : ev_n(X) = X_n \to Y$ of sets. Hence there is a bijection $\varphi_{X,Y} : \mathbf{Set}(ev_n(X), Y) \cong \mathbf{sSet}(X, (Y_n)_\bullet)$.

It remains to check that the required naturality conditions are satisfied. Let $f : W \to X$ be a morphism in **sSet** and $g : Y \to Z$ be a morphism in **Set**. Let $h : ev_n(X) \to Y$ be a morphism in **Set**. $\square$

Definition 2.7. Let $\mathcal{C}$ be a small category. The **nerve** of $\mathcal{C}$, written $N\mathcal{C}$ is the simplicial set with a 0-simplex for each object and, for $n \geq 1$, an n-simplex for each sequence of n composable morphisms, $X_0 \to X_1 \to \ldots \to X_{n-1} \to X_n$. The face morphisms are given by

$$\begin{aligned} &d_i(X_0 \to \ldots \to X_{i-1} \xrightarrow{f} X_i \xrightarrow{g} X_{i+1} \to \ldots \to X_n) \\ &= \begin{cases} (X_1 \to \ldots\ldots \to X_n) & \text{for } i = 0 \\ (X_0 \to \ldots \to X_{i-1} \xrightarrow{gf} X_{i+1} \to \ldots \to X_n) & \text{for } 0 < i < n \\ (X_1 \to \ldots \to X_{n-1}) & \text{for } i = n \end{cases} \end{aligned}$$

and the degeneracy morphisms are given by

$$\begin{aligned} &s_i(X_0 \to \ldots \to X_i \to \ldots X_n) \\ &= (X_0 \to \ldots \to X_i \xrightarrow{1_{X_i}} X_i \to \ldots \to X_n) \end{aligned}$$

it is straightforward to check that the simplicial relations are satisfied.

Chapter 3

Factorisation Systems

There are two distinct flavours of model category theoretic arguments. Those centred on the weak equivalences and those involving the co/fibrations. Arguments involving the co/fibrations generally rely on the weak factorisation systems encoded into the model category axioms. The standard references [3] and [5] prove many results about model categories that are actually results about weak factorisation systems. We prove those results in this chapter to make this distinction clear.

Definition 3.1. Let $i : A \to B$ and $p : X \to Y$ be morphisms in some category $\mathcal{M}$. We say that i has the **left lifting property** with respect to p, and that p has the **right lifting property** with respect to i if for any pair of morphisms $k : A \to X$ and $j : B \to Y$ such that $ji = pk$ there exists a morphism (called a **lift**) $B \to X$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "j"'] \arrow[ur, dashed] & Y \end{tikzcd} .$$

Let $\mathcal{I}$ be a collection of morphisms in some category $\mathcal{M}$. We write $\mathcal{I}^{\boxslash}$ for the collection of morphisms in $\mathcal{M}$ that have the right lifting property with respect to all morphisms in $\mathcal{I}$. Similarly we write ${}^{\boxslash}\mathcal{I}$ for the collection of morphisms in $\mathcal{M}$ that have the left lifting property with respect to all morphisms in $\mathcal{I}$.

Proposition 3.2. ***(The retract argument)*** *Let $\mathcal{M}$ be a category, and let $f : X \to Y$ be a morphism in $\mathcal{M}$. Then,*

(i) if $f = pi$ where p has the right lifting property with respect to f then f is a retract of i,

(ii) if $f = pi$ where i has the left lifting property with respect to f then f is a retract of p.

Proof. We show part (ii), part (i) is dual. Suppose $f = pi$ and i has the left lifting property with respect to f. Then there exists a morphism q such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ A \arrow[r, "p"'] \arrow[ur, dashed, "\exists q"] & Y \end{tikzcd} .$$

Then the diagram

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[rr, bend left, "1_X"] \arrow[d, "f"'] & A \arrow[r, "q"'] \arrow[d, "p"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

commutes with the commutativity of the top triangle, left square, and right square following from the commutativity of the top triangle, square, and bottom triangle in the previous diagram. Hence f is a retract of p. □

Proposition 3.3. *Let $\mathcal{M}$ be a category, and let $f : X \to Y$ be a morphism in $\mathcal{M}$. Then,*

(i) the class of morphisms with the left lifting property with respect to f is closed under composition,

(ii) the class of morphisms with the right lifting property with respect to f is closed under composition.

Proof. We show part (ii), part (i) is dual. Let $g : A \to B$ and $h : B \to C$ be morphisms with the right lifting property with respect to f. Suppose that j, k are morphisms such that $kf = hgj$ then as $h \in f^{\boxslash}$ there exists a lift $\varphi : Y \to B$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & A \arrow[r, "g"] & B \arrow[d, "h"] \\ Y \arrow[rr, "k"'] \arrow[urr, dashed, "\exists \varphi"] & & C \end{tikzcd} .$$

As $g \in f^{\boxslash}$ there exists a lift $\psi : Y \to A$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "f"'] & A \arrow[d, "g"] \\
Y \arrow[r, "\varphi"'] \arrow[ur, dashed, "\exists\psi"] & B
\end{tikzcd} .$$

Then we have $\psi f = j$ and $hg\psi = h\varphi = k$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[dd, "f"'] & A \arrow[d, "g"] \\
 & B \arrow[d, "h"] \\
Y \arrow[r, "k"'] \arrow[uur, "\psi"] & C
\end{tikzcd} .$$

Hence $hg \in f^{\boxslash}$. $\square$

Proposition 3.4. *Let $\mathcal{M}$ be a category, and let $f : X \to Y$ be a morphism in $\mathcal{M}$. Then,*

(i) the class of morphisms with the left lifting property with respect to f is closed under retracts and

(ii) the class of morphisms with the right lifting property with respect to f is closed under retracts.

Proof. We show part (ii), part (i) is similar. Suppose that $g : A \to B$ has the right lifting property with respect to f, and let $h : C \to D$ be a retract of g. Let p and q be morphisms such that $hp = qf$. Then we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "f"'] & C \arrow[r, "i"] \arrow[d, "h"'] \arrow[rr, bend left, "1_C"] & A \arrow[r, "j"] \arrow[d, "g"] & C \arrow[d, "h"] \\
Y \arrow[r, "q"'] \arrow[urr, dashed, "\exists\varphi"] & D \arrow[r, "k"'] \arrow[rr, bend right, "1_D"'] & B \arrow[r, "l"'] & D
\end{tikzcd}$$

where the existence of the lift φ follows from g having the right lifting property with respect to f. The commutativity of the diagram tells us that $j\varphi f = jip = 1_C p = p$ and that $q = 1_D q = lkq = hj\varphi$. Thus the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "f"'] & C \arrow[d, "h"] \\
Y \arrow[r, "q"'] \arrow[ur, "j\varphi"] & D
\end{tikzcd} .$$

Hence h has the right lifting property with respect to f. □

Proposition 3.5. *Let $\mathcal{M}$ be a category, and let $f : X \to Y$ be a morphism in $\mathcal{M}$. Then,*

(i) the class of morphisms with the left lifting property with respect to f is closed under pushouts and

(ii) the class of morphisms with the right lifting property with respect to f is closed under pullbacks.

Proof. We show part (ii), part (i) is similar. Suppose that $g : A \to B$ has the right lifting property with respect to f, and let $h : C \to B$ be a morphism such that the pullback of g along h exists and call it $p : P \to C$. Let i and k be morphisms such that $pi = kf$. Then we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & P \arrow[r, "j"] \arrow[d, "p"'] & A \arrow[d, "g"] \\ Y \arrow[r, "k"'] \arrow[urr, dashed, "\varphi"] & C \arrow[r, "h"'] & B \end{tikzcd} .$$

By the universal property of pullbacks and the commutativity of the lower triangle in the diagram there exists a unique morphism $\psi : Y \to P$ such that the following diagram commutes

$$\begin{tikzcd} Y \arrow[drr, bend left, "\varphi"] \arrow[dr, dashed, "\exists ! \psi"] \arrow[ddr, bend right, "k"'] & & \\ & P \arrow[r, "j"] \arrow[d, "p"] & A \arrow[d, "g"] \\ & C \arrow[r, "h"'] & B \end{tikzcd} .$$

Furthermore, by the commutativity of these two diagrams we have $ji = \varphi f = j\psi f$ and $hpi = hkf = hp\psi f$. Thus, by the universal property of pushouts $i = \psi f$. Hence the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & P \arrow[d, "p"] \\ Y \arrow[r, "k"'] \arrow[ur, "\psi"] & C \end{tikzcd}$$

So p has the right lifting property with respect to f. □

Definition 3.6. Let $\mathcal{M}$ be a category. A **strong factorisation system** on $\mathcal{M}$ is a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ such that:

(i) The classes $\mathcal{L}$ and $\mathcal{R}$ are closed under composition with isomorphisms.

(ii) Any morphism f in $\mathcal{M}$ factors as $f = gh$ where $g \in \mathcal{R}$ and $h \in \mathcal{L}$.

(iii) We have $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$ such that for any morphisms $i \in \mathcal{L}, p \in \mathcal{R}$ and commutative square

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "j"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

in $\mathcal{M}$ the dashed lift is unique.

Definition 3.7. Let $\mathcal{M}$ be a category. A **weak factorisation system** on $\mathcal{M}$ is a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ such that:

(F1) The classes $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

(F2) Any morphism f in $\mathcal{M}$ factors as $f = gh$ where $g \in \mathcal{R}$ and $h \in \mathcal{L}$.

(F3) We have $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Let $\mathcal{M}$ be a cocomplete category, then for any $A \in \mathrm{obj}(\mathcal{M})$ and any functor $B : \mathbb{N} \to \mathcal{M}$ (where we treat $\mathbb{N}$ as the category whose objects are the natural numbers with $\mathbb{N}(n, m) = \{*\}$ if $n \leq m$ and $\mathbb{N}(n, m) = \varnothing$ otherwise) there is a natural morphism $B(n) \to \mathrm{colim}(B)$ for each natural number n. These morphisms induce morphisms $\mathcal{M}(A, B(n)) \to \mathcal{M}(A, \mathrm{colim}(B))$ by postcomposition. These morphisms induce a canonical map $\mathrm{colim}_n(\mathcal{M}(A, B_n)) \to \mathcal{M}(A, \mathrm{colim}_n B_n)$.

Definition 3.8. An object $A \in \mathrm{obj}(\mathcal{M})$ is **sequentially small** if for all functors $B : \mathbb{N} \to \mathcal{M}$ the canonical morphism $\mathrm{colim}_n(\mathcal{M}(A, B_n)) \to \mathcal{M}(A, \mathrm{colim}_n B_n)$ is an isomorphism of sets.

The small object argument is a method of constructing a factorisation system starting from a set of morphisms. Let $I = \{g_q : A_q \to B_q\}_{q \in Q}$ be a set of morphisms in a cocomplete category $\mathcal{M}$. We will construct a (cofibrantly generated) weak factorisation system on $\mathcal{M}$ by building a factorisation for each morphism using I. Let $f : X \to Y$ be a morphism in $\mathcal{M}$, we define $S_0(q)$ to be the set of pairs (j, k) with $j, k \in \mathrm{Mor}(\mathcal{M})$ such that the following diagram commutes

$$\begin{tikzcd} A_q \arrow[r, "j"] \arrow[d, "g_q"'] & X \arrow[d, "f"] \\ B_q \arrow[r, "k"] & Y \end{tikzcd}.$$

Letting P_0 be the pushout of $\coprod_{q\in Q}\coprod_{(j,k)\in S_0(q)} g_q$ along $\coprod_{q\in Q}\coprod_{(j,k)\in S_0(q)} j$ we have the following factorisation of f.

$$\begin{tikzcd} \coprod_{q\in Q}\coprod_{(j,k)\in S_0(q)} A_q \arrow[r, "\coprod_q \coprod_{(j,k)} j"] \arrow[d, "\coprod_q \coprod_{(j,k)} g_q"'] & X \arrow[d, "i_0"] \arrow[ddr, bend left, "f"] & \\ \coprod_{q\in Q}\coprod_{(j,k)\in S_0(q)} B_q \arrow[r] \arrow[drr, bend right, "\coprod_q \coprod_{(j,k)} k"'] & P_0 \arrow[dr, dashed, "p_0"] & \\ & & Y \end{tikzcd}.$$

While this is a perfectly good factorisation of f, we would like our factorisation to have the useful property that the second morphism has the right lifting property with respect to I. For $n \in \mathbb{N}^+$, assuming P_{n-1} has been defined, we define $S_n(q)$ to be the set of pairs (j, k) with $j, k \in \mathrm{Mor}(\mathcal{M})$ such that the diagram

$$\begin{tikzcd} A_q \arrow[r, "j"] \arrow[d, "g_q"'] & P_{n-1} \arrow[d, "p_{n-1}"] \\ B_q \arrow[r, "k"] & Y \end{tikzcd}$$

commutes. Letting P_n be the pushout of $\coprod_{q\in Q}\coprod_{(j,k)\in S_n(q)} g_q$ along $\coprod_{q\in Q}\coprod_{(j,k)\in S_n(q)} j$ we have the following factorisation of f.

$$\begin{tikzcd} \coprod_{q\in Q}\coprod_{(j,k)\in S_n(q)} A_q \arrow[r, "\coprod_q \coprod_{(j,k)} j"] \arrow[d, "\coprod_q \coprod_{(j,k)} g_q"'] & P_{n-1} \arrow[d, "i_n"] \arrow[ddr, bend left, "p_{n-1}"] & \\ \coprod_{q\in Q}\coprod_{(j,k)\in S_n(q)} B_q \arrow[r] \arrow[drr, bend right, "\coprod_q \coprod_{(j,k)} k"'] & P_n \arrow[dr, dashed, "p_n"] & \\ & & Y \end{tikzcd}.$$

Then for each $n \in \mathbb{N}$ we have a factorisation $f = p_n i_n \dots i_0$. Let P be the functor $\mathbb{N} \to \mathcal{M}$ with $P(n) = P_n$ and taking the morphism $n \to n+1$ to p_n for all

$n \in \mathbb{N}$. Letting $P_\infty = \operatorname{colim} P$, we have natural morphisms $i_\infty : X \to P_\infty$ and $p_\infty : P_\infty \to Y$ such that $f = p_\infty i_\infty$.

Proposition 3.9. *Let A_q be sequentially small for all $q \in Q$, then p_∞ has the right lifting property with respect to I.*

Proof. Let $g_q \in I$, and let (j, k) be a pair of morphisms in $\mathcal{M}$ such that the diagram

$$\begin{tikzcd}
A_q \arrow[r, "j"] \arrow[d, "g_q"] & P_\infty \arrow[d, "p_\infty"] \\
B_q \arrow[r, "k"] & Y
\end{tikzcd}$$

commutes. For all $n \in \mathbb{N}$ the morphism collections $\mathcal{M}(A_q, P_n)$ are sets hence $\operatorname*{colim}_n(\mathcal{M}(A_q, P_n))$ is the quotient $\coprod_{n\in\mathbb{N}} \mathcal{M}(A_q, P_n)/\sim$ where $f \sim g$ if one of them is the other postcomposed by a sequence of i_n morphisms. We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}(A_q, P_0) \arrow[d, "i_1"] \arrow[ddr] \arrow[ddrr, bend left] & & \\
\mathcal{M}(A_q, P_1) \arrow[d, "i_2"] \arrow[dr] \arrow[drr, bend left] & & \\
\vdots \arrow[d, "i_n"] & \operatorname*{colim}_n(\mathcal{M}(A_q, P_n)) \arrow[r, dashed] & \mathcal{M}(A_q, P_\infty) \\
\mathcal{M}(A_q, P_n) \arrow[d, "i_{n+1}"] \arrow[ur] \arrow[urr, bend right] & & \\
\vdots & &
\end{tikzcd}$$

where the morphisms $\mathcal{M}(A_q, P_n) \to \mathcal{M}(A_q, P_\infty)$ are postcomposition by the natural morphisms $P_n \to P_\infty$. As A_q is sequentially small $\operatorname*{colim}_n(\mathcal{M}(A_q, P_n)) \to \mathcal{M}(A_q, P_\infty)$ is a bijection. Let $j' \in \mathcal{M}(A_q, P_m)$ be a representative of the equivalence class that j maps to under this bijection. Then by the commutativity of the triangles

$$\begin{tikzcd}
\mathcal{M}(A_q, P_m) \arrow[r] \arrow[d, "i_{m+1}"] \arrow[dr] & \operatorname{colim}_n \mathcal{M}(A_q, P_n) \arrow[d] \\
\mathcal{M}(A_q, P_{m+1}) \arrow[r] & \mathcal{M}(A_q, P_\infty)
\end{tikzcd}$$

in the previous diagram, j factors as $A_q \xrightarrow{j'} P_m \xrightarrow{i_{m+1}} P_{m+1} \to P_\infty$. The pair (j', k) is in $S_{m+1}(q)$ so there is a lift $B_q \to P_{m+1}$ as shown in the following commutative diagram

$$\begin{tikzcd}
A_q \arrow[r, "j'"'] \arrow[d, "g_q"'] \arrow[rrr, bend left, "j"] & P_m \arrow[r, "i_{m+1}"] & P_{m+1} \arrow[r] \arrow[d, "p_{m+1}"] & P_\infty \arrow[d, "p_\infty"] \\
B_q \arrow[rr, "k"'] \arrow[urr, dashed] & & Y \arrow[r, equal] & Y
\end{tikzcd} .$$

This gives a lift $B_q \to P_{m+1} \to P_\infty$. Hence p_∞ has the right lifting property with respect to I. $\square$

Definition 3.10. Given a set of morphisms I in a category $\mathcal{M}$ we call a morphism a **relative I-cell complex** if it is a transfinite composition (Definition 1.15) of pushouts of morphisms in I. We denote the collection of such morphisms by I-cell.

Lemma 3.11. *Let I be a set of morphisms in a cocomplete category $\mathcal{M}$. If f is a pushout of a coproduct of morphisms in I then f is in I-cell.*

Proof. Let K be a set and $g_k : C_k \to D_k$ a morphism in I for each $k \in K$. Let λ be an ordinal isomorphic to K. Suppose f is the pushout of $\coprod g_k$ along some morphism $\coprod C_k \to X$.

$$\begin{tikzcd}
\coprod C_k \arrow[r] \arrow[d, "\coprod g_k"'] & X \arrow[d, "f"] \\
\coprod D_k \arrow[r] & Y
\end{tikzcd}$$

Let $Q : \lambda \to \mathcal{M}$ be the λ-sequence where $Q_0 = X$, with $X_\beta \to X_{\beta+1}$ as the pushout of g_β along $C_\beta \to X_\beta$, and $X_\beta = \operatorname{colim}_{\alpha<\beta}$ for limit ordinals β. The transfinite composition Q is isomorphic to f, so f is in I-cell. $\square$

Theorem 3.12. *Let $\mathcal{M}$ be a cocomplete category, and let I be a set of morphisms in $\mathcal{M}$ such that the domains of all morphisms in I are sequentially small. Then $({}^{\boxslash}(I^{\boxslash}), I^{\boxslash})$ is a weak factorisation system on $\mathcal{M}$. We say this weak factorisation system is **cofibrantly generated** by I.*

Proof. As the morphism classes are defined by lifting properties they are closed under retracts by Proposition 3.4, hence **F1** is satisfied. **F3** is immediate. It remains to show that any morphism can be factorised as a morphism in ${}^{\boxslash}(I^{\boxslash})$ followed by a morphism in $I^{\boxslash}$. By the above lemma, the i_n morphisms of the small object argument are in I-cell, so i_∞ is a transfinite composition of transfinite compositions of pushouts of morphisms in I. Hence i_∞ is in I-cell. In light of Proposition 3.9, to show that the factorisation given above can be used to satisfy **F2** it suffices to show that $I\text{-cell} \subseteq {}^{\boxslash}(I^{\boxslash})$. Let $f : X \to Y$ be a morphism in $I^{\boxslash}$ and let $g : A \to B$ be a morphism in I-cell. Then g is a transfinite composition of a λ-sequence, C, of pushouts of elements of I for some ordinal λ. Let g_β be the morphism $C_\beta \to C_{\beta+1}$ in I. As each g_β is the pushout of a morphism with the left lifting property with respect to $I^{\boxslash}$, they have the same lifting property by Proposition 3.5. Suppose j, k are morphisms such that $fj = kg$. We define a lift by transfinite induction. We have a morphism $j : C_0 = A \to X$ such that there exists a lift making the following diagram commute.

$$\begin{tikzcd}
C_0 \arrow[rr, "j"] \arrow[d, "g_0"'] & & X \arrow[d, "f"] \\
C_1 \arrow[r] \arrow[urr, dashed, "\varphi_0"] & \operatorname{colim} C = B \arrow[r, "k"'] & Y
\end{tikzcd}$$

Given a morphism $\varphi_\beta : C_\beta \to X$ such that the outside of the following diagram commutes, there exists a lift $\varphi_{\beta+1} : C_{\beta+1} \to X$ such that the whole diagram commutes.

$$\begin{tikzcd}
C_\beta \arrow[rr, "\varphi_\beta"] \arrow[d, "g_\beta"'] & & X \arrow[d, "f"] \\
C_{\beta+1} \arrow[r] \arrow[urr, dashed, "\varphi_{\beta+1}"] & \operatorname{colim} C = B \arrow[r, "k"'] & Y
\end{tikzcd}$$

If β is a limit ordinal, given a morphism $\operatorname{colim}_{\alpha<\beta} C_\alpha \to X$ such that the outside of the following diagram commutes, there exists a lift $\varphi_\beta : C_\beta \to X$ such that the whole diagram commutes.

$$\begin{tikzcd}
\operatorname{colim}_{\alpha<\beta} C_\alpha \arrow[rr] \arrow[d] & & X \arrow[d, "f"] \\
C_\beta \arrow[r] \arrow[urr, dashed, "\varphi_\beta"] & \operatorname{colim} C = B \arrow[r, "k"'] & Y
\end{tikzcd}$$

So g has the left lifting property with respect to $I^{\boxslash}$. Hence $I\text{-cell} \subseteq {}^{\boxslash}(I^{\boxslash})$. $\square$

Definition 3.13. Let $\mathcal{M}$ be a category. A **functorial factorisation** on $\mathcal{M}$ is a pair, (α, β), of functors $\alpha, \beta : \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{2}}$ such that for all morphisms $f : X \to Y$ we have $f = (\beta f)(\alpha f)$.

Definition 3.14. Let γ be a cardinal and α be an ordinal. We say α is **γ-filtered** if it is a limit ordinal and if $A \subset \alpha$ such that $|A| \leq \gamma$, then $\sup(A) < \alpha$.

Definition 3.15. Let $\mathcal{M}$ be a cocomplete category, and let I be a class of morphisms in $\mathcal{M}$. An object A in $\mathcal{M}$ is **κ-small relative to** I for κ a cardinal if, for all κ-filtered ordinals λ and all λ-sequences X such that each morphism $X_\beta \to X_{\beta+1}$ is in I for $\beta + 1 < \lambda$, the morphism of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{M}(A, X_\beta) \to \mathcal{M}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. If there exists a cardinal κ such that an object A of $\mathcal{M}$ is κ-small relative to I then we say that A is **small relative to** I. If A is small relative to $\mathcal{M}$ we say it is **small**.

Remark 3.16. The small object argument presented above and Theorem 3.12 can be generalised so that it is sufficient that the domains of the morphisms in the generating set I are small relative to the relative I-cell complexes. That is, we have a factorisation $f = p_\infty i_\infty$ constructed similarly and $({}^{\boxslash}(I^{\boxslash}), (I^{\boxslash}))$ is a weak factorisation system on $\mathcal{M}$. In this case and the previous case, the factorisation $f = p_\infty i_\infty$ from the small object argument is a functorial factorisation. See Theorem 2.1.14 on page 32 of [5] for details.

Chapter 4

Model Categories

Model categories were first defined by Quillen in [8]. The definition we use here is the one found in [5] and [3]. This is a modification of what Quillen defined as a closed model category. In particular requiring all bicompleteness rather than just finite limits and colimits, and that the factorisations of the factorisation axiom are functorial. The second of these is not included in the definition given in [1], but in practice model categories satisfying this definition can generally be given a functorial factorisation.

Definition 4.1. A **model structure** on a bicomplete category, $\mathcal{M}$, consists of three distinguished classes of morphisms, $\mathcal{W}, \mathcal{C}, \mathcal{F}$, called the **weak equivalences**, **cofibrations**, and **fibrations** respectively satisfying the following axioms;

1. **(Two out of Three Axiom)** If f and g are composable and two of f, g, and gf are in $\mathcal{W}$ then so is the third. We say that W has the two out of three property.

2. **(Retract Axiom)** The morphism classes $\mathcal{W}, \mathcal{C}, \mathcal{F}$ are each closed under retracts.

3. **(Lifting Axiom)** The morphisms in $\mathcal{C}$ have the left lifting property with respect to the morphisms in $\mathcal{W} \cap \mathcal{F}$. The morphisms in $\mathcal{F}$ have the right lifting property with respect to the morphisms in $\mathcal{W} \cap \mathcal{C}$.

4. **(Factorisation Axiom)** There are two functorial factorisations (α, β) and (γ, δ) on $\mathcal{M}$ such that for $f : X \to Y$ in $\mathcal{M}$ we have $\alpha f \in \mathcal{C}, \beta f \in \mathcal{W} \cap \mathcal{F}, \gamma f \in \mathcal{W} \cap \mathcal{C}$, and $\delta f \in \mathcal{F}$.

A **model category** is a bicomplete category, $\mathcal{M}$, together with a model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ on $\mathcal{M}$.

A morphism in a model category which is both a weak equivalence and a cofibration is called an **acyclic cofibration**, similarly a morphism in a model category which is both a weak equivalence and a fibration is called an **acyclic fibration**.

Remark 4.2. A given category can have more than one model structure on it. We will see this when we get to examples of model categories.

Examples 4.3. (i) A trivial example of a model category is the one object category where the only morphism is the identity morphism which is a weak equivalence, a cofibration, and a fibration.

(ii) Less trivially let $\mathcal{M}$ be a bicomplete category. There is a model structure on $\mathcal{M}$ given by letting $\mathcal{W}$ be the class of all isomorphisms and letting $\mathcal{C}$ and $\mathcal{F}$ be the class of all morphisms in $\mathcal{M}$. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{M}$. The two out of three axiom is obviously satisfied for isomorphisms. As they contain all morphisms $\mathcal{C}$ and $\mathcal{F}$ are clearly closed under retracts. If $g : X \to Y$ is an isomorphism and $f : A \to B$ is a retract of g then there exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "1_A"] & X \arrow[r, "j"'] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, "k"] \arrow[rr, bend right, "1_B"'] & Y \arrow[r] & B
\end{tikzcd} .$$

Then $f(jg^{-1}k) = 1_B$ and $(jg^{-1}k)f = 1_A$ so f is an isomorphism with inverse $jg^{-1}k$. So $\mathcal{W}$ is closed under retracts. Hence the retract axiom is satisfied. Given the solid arrows in the diagram commute

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "k"'] \arrow[ur, dashed, "\varphi"] & Y
\end{tikzcd}$$

if f is an isomorphism then putting $\varphi = jf^{-1}$ the above diagram commutes, similarly if g is an isomorphism then putting $\varphi = g^{-1}k$ makes the diagram commute. Hence the lifting axiom is satisfied. Finally the required functorial factorisations are given by $\alpha f = f$, $\beta f = 1_B$, $\gamma f = 1_A$, and $\delta f = f$.

For less trivial examples, showing that a given category $\mathcal{M}$ together with morphism classes $\mathcal{W}$, $\mathcal{C}$, and $\mathcal{F}$ is a model category directly from the axioms can be quite difficult. In practice one does this by showing that a model category is a cofibrantly generated model category. Example of categories with interesting model structures include the category of compactly generated, weakly Hausdorff topological spaces, the category of simplicial sets, the category of R-modules where R is a Frobenius ring, and the category of chain complexes of modules over a ring R. We leave the description of these model structures until after we have established some basic results about model categories and have defined cofibrantly generated model categories.

Definition 4.4. Let $\mathcal{M}$ be a model category, and let X be an object in $\mathcal{M}$. We say that X is **cofibrant** if the morphism $\varnothing \to X$ from the initial object is a cofibration in $\mathcal{M}$. We say that X is **fibrant** if the morphism $X \to *$ to the final object is a fibration in $\mathcal{M}$.

Definition 4.5. Let $\mathcal{M}$ be a model category. For an object X in $\mathcal{M}$ we can apply the functorial factorisation (α, β) to the morphism from the initial object to X to get $\varnothing \to QX \xrightarrow{q_X} X$ where QX is a cofibrant object and q_X is an acyclic fibration. Let $f : X \to Y$ be a morphism in $\mathcal{M}$ then f corresponds to a unique morphism in $\mathcal{M}^2$. Applying α gives a unique morphism $Qf : QX \to QY$.

$$\alpha\left(\begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}\right) = \begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow \\ QX & \xrightarrow{Qf} & QY \end{array}$$

We define the **cofibrant replacement functor** $Q : \mathcal{M} \to \mathcal{M}$ on objects and morphisms as above. The functoriality of Q follows from the functoriality of α. The morphisms $q_X : QX \to X$ are the components of a natural transformation $q : Q \implies 1_{\mathcal{M}}$. Similarly we define the **fibrant replacement functor** by applying the functorial factorisation (γ, δ) to the morphism $X \to *$ from an object X to the final object. We get $X \xrightarrow{r_X} RX \to *$ where RX is a fibrant object and r_X is an acyclic cofibration. A morphism $f : X \to Y$ in $\mathcal{M}$ corresponds to a unique morphism in $\mathcal{M}^2$. Applying δ gives a unique morphism $Rf : RX \to RY$.

$$\alpha\left(\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}\right) = \begin{array}{ccc} RX & \xrightarrow{Rf} & RY \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

The functoriality of R follows from the functoriality of δ. The morphisms $r_X : X \to RX$ are the components of a natural transformation $r : 1_{\mathcal{M}} \implies R$.

Proposition 4.6. *Let $\mathcal{M}$ be a model category with model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$. Then,*

(i) a morphism f in $\mathcal{M}$ is a cofibration if and only if it has the left lifting property with respect to all acyclic fibrations,

(ii) a morphism f in $\mathcal{M}$ is an acyclic cofibration if and only if it has the left lifting property with respect to all fibrations,

(iii) a morphism f in $\mathcal{M}$ is a fibration if and only if it has the right lifting property with respect to all acyclic cofibrations,

(iv) a morphism f in $\mathcal{M}$ is an acyclic fibration if and only if it has the right lifting property with respect to all cofibrations.

Proof. We prove part (i), the other proofs are similar. Suppose that $f \in \mathcal{C}$ then f has the left lifting property with respect to $\mathcal{W} \cap \mathcal{F}$ by the lifting axiom. Conversely, suppose that f has the left lifting property with respect to all morphisms in $\mathcal{W} \cap \mathcal{F}$. By the factorisation axiom f factors as $X \xrightarrow{\alpha f} \alpha Y \xrightarrow{\beta f} Y$ where $\alpha f \in \mathcal{C}$ and $\beta f \in \mathcal{W} \cap \mathcal{F}$. By assumption f has the left lifting property with respect to βf. By the retract argument f is a retract of αf. Hence by the retract axiom $f \in \mathcal{C}$. $\square$

Remark 4.7. Proposition 4.6 together with the retract and factorisation axioms show that in any model category $\mathcal{M}$ the pairs $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ are weak factorisation systems on $\mathcal{M}$.

Proposition 4.8. *Let $\mathcal{M}$ be a model category with model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$. Then a morphism f in $\mathcal{M}$ is a weak equivalence if and only if $f = pi$ where p is an acyclic fibration and i is an acyclic cofibration.*

Proof. Suppose f is a weak equivalence. By the factorisation axiom we can factorise f as $f = (\beta f)(\alpha f)$ where $\alpha f \in \mathcal{C}$ and $\beta f \in \mathcal{W} \cap \mathcal{F}$. By the two out of three axiom αf is a weak equivalence, hence $\alpha f \in \mathcal{W} \cap \mathcal{C}$. The converse is immediate by the two out of three axiom. $\square$

Proposition 4.9. *Let $\mathcal{M}$ be a model category. Any two of the classes of weak equivalences, cofibrations, and fibrations determines the other.*

Proof.

- If the weak equivalences and cofibrations are known then by Proposition 4.6 part (ii) the fibrations are precisely those morphisms with the right lifting property with respect to the acyclic cofibrations.

- If the weak equivalences and fibrations are known then by Proposition 4.6 part (iv) the cofibrations are precisely those morphisms with the right lifting property with respect to the acyclic fibrations.

- If the cofibrations and fibrations are known then by Proposition 4.6 parts (i) and (iii) the acyclic cofibrations and acyclic fibrations are known. By Proposition 4.8 the weak equivalences are precisely those morphism which can be written as an acyclic cofibration followed by an acyclic fibration.

□

Functors between model categories which preserve the model structures, called Quillen functors, are particularly important because they induce functors between the homotopy categories. When the induced functor is an equivalence of categories these are called Quillen equivalences and can allow the same homotopy category to be studied using model categories that are not equivalent.

Definition 4.10. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : G$ be an adjoint pair of functors where $F \dashv G$. We say that F is a **left Quillen functor** if F preserves cofibrations and acyclic cofibrations. We say that G is a **right Quillen functor** if G preserves fibrations and acyclic fibrations. If (F, G, φ) is an adjunction and F is a left Quillen functor we call it a **Quillen adjunction**.

Proposition 4.11. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : G$ be an adjoint pair of functors where F is left adjoint to G. Then F is a left Quillen functor if and only if G is a right Quillen functor.*

Proof. Suppose F is a left Quillen functor. Let $p : X \to Y$ be a fibration in $\mathcal{N}$ and $i : A \to B$ be an acyclic cofibration in $\mathcal{M}$. By the adjunction $F \dashv G$ the first diagram (in $\mathcal{M}$)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & GX \arrow[d, "Gp"] \\ B \arrow[r] \arrow[ur, dashed] & GY \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r] \arrow[d, "Fi"'] & X \arrow[d, "p"] \\ FB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has the lift shown if and only the second diagram (in $\mathcal{N}$) has the lift shown. As F is a left Quillen functor Fi is an acyclic cofibration in $\mathcal{N}$, so Fi has the left lifting property with respect to p. In particular the second diagram above has the lift shown and hence so does the first. Hence Gp has the right lifting property with respect to all acyclic cofibrations in $\mathcal{M}$, that is Gp is a fibration in $\mathcal{M}$. Hence G preserves fibrations. Replacing acyclic cofibration and fibration with cofibration and acyclic fibration above shows that G preserves acyclic fibrations. Hence G is a right Quillen functor. A similar argument shows that if G is a right Quillen functor then F is a left Quillen functor. $\square$

Definition 4.12. Let $\mathcal{M}$ be a category with a distinguished class of morphisms, $\mathcal{W}$ called weak equivalences. The **homotopy "category"**, $\operatorname{Ho}\mathcal{M}$, is the "category" we get by formally adding inverses for the weak equivalences. More specifically, let $F(\mathcal{M}, \mathcal{W}^{-1})$ be the free category generated by $\mathcal{M}$ and a formal inverse w^{-1} for each morphism $w \in \mathcal{W}$ where if $w : X \to Y$ then $w^{-1} : Y \to X$. The objects of this category are the objects of $\mathcal{M}$ and the morphisms are finite strings $(f_1, \ldots, f_n)$ of composable morphisms where each f_i is either a morphism in $\mathcal{M}$ or is w^{-1} for some $w \in \mathcal{W}$. Composition is concatenation of strings and the empty string at an object is the identity morphism for that object. Then $\operatorname{Ho}\mathcal{M}$ is the quotient category of $F(\mathcal{M}, \mathcal{W}^{-1})$ by the relations that for all $X \in \operatorname{obj}(\mathcal{M})$ we have $1_X = (1_X)$, for all composable arrows of $\mathcal{M}$ we have $(f, g) = (gf)$, and for $w : X \to Y$ in $\mathcal{W}$ we have $(w, w^{-1}) = 1_X$ and $(w^{-1}, w) = 1_Y$.

Remark 4.13. As defined $\operatorname{Ho}\mathcal{M}$ is not necessarily a category. In particular, $\operatorname{Ho}\mathcal{M}(X, Y)$ might be a proper class. Part of the motivation for studying model categories is that if $\mathcal{M}$ is a model category where $\mathcal{W}$ are the weak equivalences of the model structure then $\operatorname{Ho}\mathcal{M}$ is a category. This is shown in Theorem 1.2.10 on page 13 of [5].

Formal inversion of a class of morphisms in this way is called localisation. There is an obvious inclusion functor $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ taking objects and morphisms to themselves, in particular taking weak equivalences to isomorphisms. This functor has the universal property[1] that if $\varphi : \mathcal{M} \to \mathcal{N}$ is a functor such that $\varphi(w)$ is an isomorphism whenever w is a weak equivalence, then there exists a unique functor $\delta : \operatorname{Ho}\mathcal{M} \to \mathcal{N}$ such that $\varphi = \delta\gamma$.

Definition 4.14. Let $\mathcal{M}$ be a model category. We write $\mathcal{M}_c$, $\mathcal{M}_f$, and $\mathcal{M}_{cf}$ for the full subcategories of $\mathcal{M}$ whose objects are respectively, the cofibrant objects,

[1]See Lemma 1.2.2 on page 7 of [5] for a proof.

the fibrant objects, and objects which are both cofibrant and fibrant. If w was a weak equivalence in $\mathcal{M}$ we will still consider it so in these subcategories.

Proposition 4.15. *Let $\mathcal{M}$ be a model category. By the universal property of the inclusion functors of categories into their homotopy categories, the inclusions functors $\mathcal{M}_{cf} \hookrightarrow \mathcal{M}_c \hookrightarrow \mathcal{M}$ and $\mathcal{M}_{cf} \hookrightarrow \mathcal{M}_f \hookrightarrow \mathcal{M}$ induce functors $\operatorname{Ho}\mathcal{M}_{cf} \to \operatorname{Ho}\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}$ and $\operatorname{Ho}\mathcal{M}_{cf} \to \operatorname{Ho}\mathcal{M}_f \to \operatorname{Ho}\mathcal{M}$. These induced functors are equivalences of categories.*

Proof. We show that the functor $\operatorname{Ho}\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}$ induced by the inclusion $\iota : \mathcal{M}_c \hookrightarrow \mathcal{M}$ is an equivalence, the other cases are similar. A morphism w in $\mathcal{M}_c$ is only a weak equivalence if it is a weak equivalence in $\mathcal{M}$ so weak equivalences go to isomorphisms under the functor $\mathcal{M}_c \overset{\iota}{\hookrightarrow} \mathcal{M} \to \operatorname{Ho}\mathcal{M}$. So by the universal property of $\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}_c$ there is a functor $\operatorname{Ho}\iota : \operatorname{Ho}\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}$ induced by ι. The cofibrant replacement functor $Q : \mathcal{M} \to \mathcal{M}$ takes objects in $\mathcal{M}$ to objects in $\mathcal{M}_c$ and so we can consider it a functor $\mathcal{M} \to \mathcal{M}_c$. If $f : X \to Y$ is a weak equivalence then we have the commutative diagram

$$\begin{tikzcd} QX \arrow[r, "Qf"] \arrow[d, "q_X"] & QY \arrow[d, "q_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{M}$ where q_X and q_Y are weak equivalences. So by the two out of three axiom Qf is a weak equivalence. So weak equivalences go to isomorphisms under the functor $\mathcal{M} \xrightarrow{Q} \mathcal{M}_c \to \operatorname{Ho}\mathcal{M}_c$. So by the universal property of $\mathcal{M} \to \operatorname{Ho}\mathcal{M}$ there is a functor $\operatorname{Ho}Q : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{M}_c$ induced by Q. For all objects X in $\mathcal{M}$ the morphism q_X is a weak equivalence $\iota Q(X) = QX \to X$ and so are isomorphisms $QX \to X$ in $\operatorname{Ho}\mathcal{M}$. These give a natural isomorphism $(\operatorname{Ho}\iota)(\operatorname{Ho}Q) \implies 1_{\mathcal{M}}$. Considering only the objects in $\mathcal{M}_c$ in the same way gives a natural isomorphism $(\operatorname{Ho}Q)(\operatorname{Ho}\iota) \implies 1_{\mathcal{M}}$. Hence $\operatorname{Ho}\iota$ is an equivalence of categories. □

Definition 4.16. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories.

(i) If $F : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor the **total left derived functor**, $LF : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{N}$ is the composition

$$\operatorname{Ho}\mathcal{M} \xrightarrow{\operatorname{Ho}Q} \operatorname{Ho}\mathcal{M}_c \xrightarrow{\operatorname{Ho}F} \operatorname{Ho}\mathcal{N}.$$

(ii) If $G : \mathcal{N} \to \mathcal{M}$ is a right Quillen functor the **total right derived functor**, $RG : \operatorname{Ho}\mathcal{N} \to \operatorname{Ho}\mathcal{M}$ is the composition

$$\operatorname{Ho}\mathcal{N} \xrightarrow{\operatorname{Ho}R} \operatorname{Ho}\mathcal{N}_f \xrightarrow{\operatorname{Ho}G} \operatorname{Ho}\mathcal{M}.$$

Lemma 4.17. *Let (F, G, φ) be a Quillen adjunction with $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$. Then there is an adjuntion $L(F, G, \varphi) = (LF, RG, R\varphi)$ called the **derived adjunction**.*

For a proof that the derived adjunction is an adjunction see Lemma 1.3.10 on page 18 of [5].

Definition 4.18. A Quillen adjunction (F, G, φ) with $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ is a **Quillen equivalence** if for all cofibrant objects X in $\mathcal{M}$ and all fibrant objects Y in $\mathcal{N}$, a morphism $f : FX \to Y$ is a weak equivalence if and only if $\varphi(f) : X \to GY$ is a weak equivalence.

Proposition 4.19. *Let (F, G, φ) be a Quillen adjunction with $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$. Then the following are equivalent:*

(i) *(F, G, φ) is a Quillen equivalence.*

(ii) *The composition*

$$X \xrightarrow{\eta_X} GFX \xrightarrow{G_{r_{FX}}} GRFX$$

is a weak equivalence of all cofibrant objects X in $\mathcal{M}$, and the composition

$$FQGY \xrightarrow{F_{q_{GY}}} FGY \xrightarrow{\varepsilon_Y} Y$$

is a weak equivalence for all fibrant objects Y in $\mathcal{N}$.

(iii) *The derived adjunction $L(F, G, \varphi)$ is an adjoint equivalence of categories.*

Proof. See Proposition 1.3.13 on page 19 of [5]. □

Definition 4.20. A **cofibrantly generated model category**, $\mathcal{M}$, is a model category such that there exist sets of morphisms I, J called the **generating cofibrations** and the **generating acyclic cofibrations** respectively such that the domains of I and J are small relative to I-cell and J-cell respectively, and the weak factorisation systems from Remark 4.7 are those cofibrantly generated by I and J. That is, $(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) = ({}^{\boxslash}(I^{\boxslash}), I^{\boxslash})$ and $(\mathcal{W} \cap \mathcal{C}, \mathcal{F}) = ({}^{\boxslash}(J^{\boxslash}), J^{\boxslash})$

Theorem 4.21. *Let $\mathcal{M}$ be a bicomplete category, let $\mathcal{W}$ be a class of morphisms in $\mathcal{M}$, and let I, J be sets of morphisms in $\mathcal{M}$. Then there is a model structure on $\mathcal{M}$ where $\mathcal{W}$ is the class of weak equivalences, I is the set of generating cofibrations, and J is the set a generating acyclic cofibrations if and only if the following conditions hold:*

(i) *The class $\mathcal{W}$ has the two out of three property and is closed under retracts*

(ii) *The domains of elements of I are small relative to I-cell*

(iii) *The domains of elements of J are small relative to J-cell*

(iv) *The class J-cell is contained in $\mathcal{W} \cap {}^{\boxslash}(I^{\boxslash})$*

(v) *The class $I^{\boxslash}$ is contained in $\mathcal{W} \cap J^{\boxslash}$*

(vi) *Either the class $\mathcal{W} \cap {}^{\boxslash}(I^{\boxslash})$ is contained in ${}^{\boxslash}(J^{\boxslash})$ or the class $\mathcal{W} \cap J^{\boxslash}$ is contained in $I^{\boxslash}$.*

Proof. Suppose that $\mathcal{M}$ is a cofibrantly generated model category where I, J are the generating cofibrations and the generating acyclic cofibrations respectively. By the two out of three axiom condition (i) is satisfied. The smallness conditions (ii) and (iii) are satisfied by definition. If $f \in J$-cell then by Theorem 3.12 $f \in {}^{\boxslash}(J^{\boxslash}) = \mathcal{W} \cap \mathcal{C} = \mathcal{W} \cap {}^{\boxslash}(I^{\boxslash})$ so condition (iv) is satisfied. Condition (v) is satisfied as $I^{\boxslash} = \mathcal{W} \cap \mathcal{F} = \mathcal{W} \cap J^{\boxslash}$. Both conditions in part (vi) are satisfied as $\mathcal{W} \cap {}^{\boxslash}(I^{\boxslash}) = \mathcal{W} \cap \mathcal{C} = {}^{\boxslash}(J^{\boxslash})$ and $\mathcal{W} \cap J^{\boxslash} = \mathcal{W} \cap \mathcal{C} = I^{\boxslash}$.

Conversely suppose that conditions (i) - (vi) are satisfied. By condition (i) the two out of three axiom is satisfied. Define the class of cofibrations $\mathcal{C}$ to be ${}^{\boxslash}(I^{\boxslash})$ and define the class of fibrations $\mathcal{F}$ to be $J^{\boxslash}$. As these are defined by lifting properties they are closed under retracts by Proposition 3.4. Together with condition (i), this shows that the retract axiom is satisfied.

By Theorem 3.12 and Remark 3.16, conditions (ii) and (iii) allow us to apply the small object argument using I and J to get functorial factorisations (α, β) and (γ, β) such that for all morphisms f in $\mathcal{M}$, $\alpha(f)$ is in ${}^{\boxslash}(I^{\boxslash}) = \mathcal{C}$, $\beta(f)$ is in $I^{\boxslash} \subseteq \mathcal{W} \cap J^{\boxslash} = \mathcal{W} \cap \mathcal{F}$ by condition (v), $\gamma(f)$ is in J-cell $\subseteq \mathcal{W} \cap J^{\boxslash} = \mathcal{W} \cap \mathcal{C}$ by condition (iv), and $\delta(f)$ is in $J^{\boxslash} = \mathcal{F}$. So the factorisation axiom is satisfied.

Finally we need to show that the lifting axiom holds. Suppose that $\mathcal{W} \cap {}^{\boxslash}(I^{\boxslash}) = \mathcal{W} \cap \mathcal{C}$ is contained in ${}^{\boxslash}(J^{\boxslash}) = {}^{\boxslash}\mathcal{F}$, that is every acyclic cofibration has the left lifting property with respect to every fibration. Suppose p is an acyclic fibration, we have a functorial factorisation $p = (\beta p)(\alpha p)$ where βp is in $I^{\boxslash} \subseteq \mathcal{W} \cap \mathcal{F}$, and αp is in $\mathcal{C}$. As the weak equivalences satisfy the two out of three property αp is also in $\mathcal{W}$. As p is a fibration it has the right lifting property with respect to αp. By the retract argument (Proposition 3.2 (ii)) p is a retract of βp. So by Proposition 3.4 p is in $I^{\boxslash}$ and so has the right lifting property with respect to every cofibration. So the lifting axiom is satisfied. Alternatively, suppose that $\mathcal{W} \cap J^{\boxslash} = \mathcal{W} \cap \mathcal{F}$ is contained in $I^{\boxslash}$. Then every acyclic fibration has the right lifting property with respect to every cofibration (morphisms in ${}^{\boxslash}(I^{\boxslash}) = \mathcal{C}$). Suppose i is an acyclic cofibration, we have a functorial factorisation $i = (\delta i)(\gamma i)$ where δi is in $\mathcal{F} = J^{\boxslash}$ and γi is in J-cell $\subseteq \mathcal{W} \cap \mathcal{C}$. Recall from the proof of Theorem 3.12 that J-cell $\subseteq {}^{\boxslash}(J^{\boxslash})$. As the weak equivalences satisfy the two out of three axiom δi is also in $\mathcal{W}$. As i is a cofibration it has the left lifting property with respect to δi. By the retract argument (Proposition 3.2 (i)) i is a retract of γi. So by Proposition 3.4 i has the left lifting property with respect to the fibrations because γi does. Hence the lifting axiom is satisfied. $\square$

The roots of abstract homotopy theory lie in algebraic topology. An important model structure then is the classic model structure on the category of topological spaces. The category of topological spaces and continuous functions between them, **Top**, is a poor category to work with as the product functor $- \times X : \mathbf{Top} \to \mathbf{Top}, W \mapsto W \times X$ does not commute with colimits for general X. In the following we consider S^{-1} to be the empty set.

The following proof that the category of compactly generated, weakly Hausdorff, topological spaces is a cofibrantly generated model category mostly follows the proof in [5]. However the proof that condition (vi) is satisfied provided here is more elementary than the proof in [5] (Theorem 2.4.12).

Theorem 4.22. *The category, $\mathcal{T}$ of compactly generated, weakly Hausdorff, topological spaces is a cofibrantly generated model category where the generating cofibrations are the set of boundary inclusions $I' = \{S^{n-1} \hookrightarrow D^n\}_{n\in\mathbb{N}}$, the generating acyclic cofibrations are the set of inclusions $J = \{D^n \hookrightarrow D^n \times I, x \mapsto (x, 0)\}_{n\in\mathbb{N}}$, and the class of weak equivalences $\mathcal{W}$ consists those morphisms $f : X \to Y$ which induce a bijection of path components and an isomorphism of homotopy groups $f_* : \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(Y, f(x_0))$ for all $n \geq 1$ and all choices of basepoint $x_0 \in X$.*

Proof. We prove this by showing that the conditions of Theorem 4.21 hold.

(i) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{T}$. Clearly if any two of f, g, and gf induce a bijection of path components so does the third. Let $n \in \mathbb{N}$. If $\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0))$ and $\pi_n(Y, y_0) \xrightarrow{g_*} \pi_n(Z, g(y_0))$ are isomorphisms then clearly $\pi_n(X, x_0) \xrightarrow{(gf)_*} \pi_n(Z, g(f(x_0)))$ is an isomorphism. Similarly, if g_* and $(gf)_*$ are isomorphisms it is immediate that f_* is an isomorphism. Suppose $\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0))$ and $\pi_n(X, x_0) \xrightarrow{(gf)_*}$

$\pi_n(Z, g(f(x_0)))$ are isomorphisms for all $x_0 \in X$, and that f and gf induce bijections of path components. If there is some $x_0 \in X$ such that $f(x_0) = y_0$ then clearly $\pi_n(Y, y_0) \xrightarrow{g_*} \pi_n(Z, g(y_0))$ is an isomorphism. If y_0 is not in the image of f, by the assumption that f induces a bijection on path components there exists some x_0 such that $f(x_0)$ is in the same path component as y_0. So there is a path $p : I \to Y$ such that $p(0) = f(x_0)$ and $p(1) = y_0$, as g is continuous $gp : I \to Z$ is a path from $g(f(x_0))$ to $g(y_0)$. By change of basepoint isomorphisms β_p and β_{gp} (see [2] Proposition 1.5) there is a commutative diagram as follows.

$$\begin{tikzcd}
\pi_n(Y, y_0) \arrow[r, "g_*"] \arrow[d, "\beta_p"] & \pi_n(Z, g(y_0)) \arrow[d, "\beta_{gp}"] \\
\pi_n(Y, f(x_0)) \arrow[r, "g_*"] & \pi_n(Z, g(f(x_0)))
\end{tikzcd}$$

As the other three morphisms are isomorphisms so is the top one. Hence $\mathcal{W}$ has the two out of three property.

(ii) Given a space X_0 in $\mathcal{T}$ the pushout of a boundary inclusion $S^{n-1} \hookrightarrow D^n$ along a morphism $\phi : S^{n-1} \to X_0$ glues an n-cell to X_0 along the attaching map ϕ. So if $X : \mathbb{N} \to \mathcal{T}$ is an $\mathbb{N}$-sequence in I'-cell then (X_{n+1}, X_n) is a relative CW complex for all $n \in \mathbb{N}$. In particular $\operatorname{colim}_n X_n = \bigcup_n X_n$. Given a morphism $f : S^m \to \operatorname{colim}_n X_n$, as S^m is compact the image $f(S^m)$ intersects the interior of at most finitely many cells so there exists $j \in \mathbb{N}$ such that f factors through X_j by $S^m \xrightarrow{f} X_j \hookrightarrow \operatorname{colim}_n X_n$. So the canonical morphism $\operatorname*{colim}_n(\mathcal{T}(S_m, X_n)) \to \mathcal{T}(S^m, \operatorname*{colim}_n X_n)$ is a bijection of sets for all $m \in \mathbb{N}$. Hence the domains of the I' morphisms are sequentially small relative to I'-cell.

(iii) Given a space X_0 in $\mathcal{T}$ the pushout of an inclusion $D^n \hookrightarrow D^n \times I, x \mapsto (x, 0)$ along a morphism $\phi : D^n \to X_0$ glues the finite CW complex $D^n \times I$ along an n-disk. So if $X : \mathbb{N} \to \mathcal{T}$ is a $\mathbb{N}$-sequence in J-cell then (X_{n+1}, X_n) is a relative CW complex for all $n \in \mathbb{N}$. In particular $\operatorname{colim}_n X_n = \bigcup_n X_n$. Given a morphism $f : D^m \to \operatorname{colim}_n X_n$, as D^m is compact the image $f(D^m)$ intersects the interior of at most finitely many cells so there exists $j \in \mathbb{N}$ such that f factors through X_j by $D^m \xrightarrow{f} X_j \hookrightarrow \operatorname{colim}_n X_n$. So the canonical morphism $\operatorname*{colim}_n(\mathcal{T}(D_m, X_n)) \to \mathcal{T}(D^m, \operatorname*{colim}_n X_n)$ is a bijection of sets for all $m \in \mathbb{N}$. Hence the domains of the J morphisms are sequentially small relative to J-cell.

(iv) Morphisms in J-cell are transfinite compositions of inclusion maps $X_n \hookrightarrow X_{n+1}$ where (X_{n+1}, X_n) is a relative CW complex for all n. In particular X_{n+1} is always X_n with a finite CW complex $D_m \times I$ attached along some m-disk, so X_n is a deformation retract of X_{n+1}. So if $f : X \to Y$ is in J-cell then f is a homotopy equivalence, hence $f \in \mathcal{W}$.

Let $f : X \to Y$ be a morphism in J-cell. Then f is some transfinite composition of pushouts of morphisms of the form $D^n \hookrightarrow D^n \times I, x \mapsto (x, 0)$. As per (iii) a pushout of such a morphism along some $D^n \to X$ is a gluing of a finite CW complex $D^n \times I$ to X. In particular $D^n \times I$ has 9 cells and so a pushout of a J morphism is a composition of pushouts of I' morphisms. Hence f is a transfinite composition of pushouts of I' morphisms. Hence J-cell $\subseteq I'$-cell $\subseteq {}^{\boxslash}((I')^{\boxslash})$.

(v) Let $p : X \to Y$ be a morphism with the right lifting property with respect to I' morphisms. Let $i : D^n \to D^n \times I$ be a J morphism, and let j and k be morphisms such that $pj = ki$. As mentioned above, $D^n \times I$ is obtained from D^n by attaching finitely many cells. So a J morphism is a finite composition of morphisms $D^n \xrightarrow{\varphi_1} C_1 \to \ldots \to C_{m-1} \xrightarrow{\varphi_m} C_m = D^n \times I$ where each φ_k is a pushout of an I' morphism. As I' morphisms have the left lifting property with respect to p, by Proposition 3.5 so does each φ_k. We have a morphism $j : D^n \to X$ such that there exists a lift making the following diagram commute.

$$\begin{tikzcd}
D^n \arrow[rr, "j"] \arrow[d, "\varphi_1"'] & & X \arrow[d, "p"] \\
C_1 \arrow[r] \arrow[urr, dashed, "\psi_1"] & D^n \times I \arrow[r, "k"'] & Y
\end{tikzcd}$$

Given a morphism $\psi_{k-1} : C_{k-1} \to X$ such that the outside of the following diagram commutes, there exists a lift $\psi_k : C_k \to X$ such that the whole diagram commutes.

$$\begin{tikzcd}
C_{k-1} \arrow[rr, "\psi_{k-1}"] \arrow[d, "\varphi_k"'] & & X \arrow[d, "p"] \\
C_k \arrow[r] \arrow[urr, dashed, "\psi_k"] & D^n \times I \arrow[r, "k"'] & Y
\end{tikzcd}$$

Hence a finite induction gives us a lift $\psi = \psi_m$ such that the following diagram commutes. Hence p has the right lifting property with respect to the J morphisms.

$$\begin{tikzcd} D^n \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ D^n \times I \arrow[r, "k"'] \arrow[ur, "\psi"] & Y \end{tikzcd}$$

Consider the homomorphism $p_* : \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ of homotopy group induced by p for some $n \in \mathbb{N}$. Suppose that $p_*([f]) = 0$ for some $f : S^n \to X$. As p has the right lifting property with respect to I' morphisms there is a lift such that the following diagram commutes.

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ D^{n+1} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

So f is null-homotopic. Thus, $[f] = 0$ so p_* is injective. The n sphere S^n is a CW complex, in particular it can be constructed from the empty space by attached finitely many cells. So the morphism $\varnothing \to S^n$ has the left lifting property with respect to $I^{\boxslash}$. So given a morphism $g : S^n \to Y$ there exists a lift such that the following diagram commutes.

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ S^n \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

So p_* is surjective. Hence p is a weak equivalence.

(vi) Let $p : X \to Y$ be a morphism in $\mathcal{W} \cap J^{\boxslash}$. For $n \in \mathbb{N}$ suppose there exist morphisms j and k such that the following diagram commutes.

$$\begin{tikzcd} S^{n-1} \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ D^n \arrow[r, "k"'] & Y \end{tikzcd}$$

As p is a weak equivalence the induced morphism $p_* : \pi_{n-1}(X, x_0) \to \pi_{n-1}(Y, p(x_0))$ is an isomorphism. The composition ki is null-homotopic, so by the isomorphism $[j] = 0$ in $\pi_{n-1}(X, x_0)$. Hence j extends to a morphism $\overline{f} : D^n \to X$. We can glue two copies of D^n together to get S^n as the pushout of i along itself. This gives us a morphism $\overline{g} : S^n \to Y$ from the following commutative diagram.

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[d, "i"'] & D^n_+ \arrow[r, "\overline{f}"] \arrow[d] & X \arrow[dd, "p"] \\
D^n_- \arrow[r] \arrow[drr, bend right, "g"'] & S^n \arrow[dr, dashed, "\exists! \overline{g}"] & \\
 & & Y
\end{tikzcd}$$

Here the subscripts on the copies of D^n are to distinguish them. As $p_* : \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ is an isomorphism there exists a lift $\tilde{f} : S^n \to X$ in the above diagram together with a homotopy $H : S^n \times I \to Y$ between $p\tilde{f}$ and $\overline{g}$. We can choose H so that $H|_{D^n_+ \times \{t\}} = \overline{f}$ for all $t \in I$. With the morphisms constructed we have the following commutative diagram.

$$\begin{tikzcd}
(S^n \times \{0\}) \cup (D^n_+ \times I) \arrow[r, "\tilde{f} \cup (\overline{f} \times I)"] \arrow[d] & X \arrow[d, "p"] \\
S^n \times I \arrow[r, "H"'] & Y
\end{tikzcd}$$

The right hand morphism is a relative J-cell complex. As p is in $J^{\boxslash}$ there exists a lift $\tilde{H} : S^n \times I \to X$ in the above diagram. Put $\tilde{h} = \tilde{H}|_{S^n \times \{1\}} : S^n \to X$. Then $\tilde{h}|_{D^n_+} = \overline{f}$ and $p\tilde{h}|_{D^n_-} = \overline{g}|_{D^n_-} = g$. So $\tilde{h}|_{D^n_-}$ is a lift in the original diagram. Hence p is in $I^{\boxslash}$.

□

For a proof of the following proposition see Proposition 1.1.8 on page 5 of [5].

Proposition 4.23. *Let $\mathcal{M}$ be a model category. There is a model structure on $\mathcal{M}_+$ where a morphism f is a weak equivalence (cofibration, fibration) if and only if Uf is a a weak equivalence (cofibration, fibration) in $\mathcal{M}$, where U is the forgetful functor (see Defintion 1.20).*

Corollary 4.24. *The category $\mathcal{T}_+$ of compactly generated, weak Hausdorff, based topological spaces is a cofibrantly generated model category where the weak equivalences are those determined be the previous theorem and lemma and the generating cofibrations, I'_+, and generating acyclic cofibrations, J_+, are the based analogues of the I' and J morphisms from the previous theorem.*

The n**-dimensional simplex** in $\mathcal{T}$ is

$$\Delta^n_{\mathcal{T}} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} = 1, x_i \geq 0 \text{ for all } i\}.$$

There is an adjoint pair of functors $|\cdot| : \mathbf{sSet} \rightleftarrows \mathcal{T} : S_\bullet$. The functor $|\cdot|$ is called **geometric realisation**. The geometric realisation $|X|$ of a simplicial set X is a topological space with one n-dimensional simplex for each n simplex with the jth $(n-1)$-dimensional face

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1, x_i \geq 0 \text{ for all } i, x_j = 0\}$$

of an n-dimensional simplex being identified with the $(n-1)$-dimensional simplex corresponding to by the $(n-1)$ simplex mapped to under the jth face morphism. For the representables in **sSet** we define $|\Delta^n| = \Delta^n_{\mathcal{T}}$. Now we can explicitly define $|X|$ for some simplicial set X, treating X_n as a topological space by giving it the discrete topology we have $|X| = \coprod_n (|\Delta^n| \times X_n)/\sim$ where $(f(\sigma), x) \sim (\sigma, f(x))$ for all $f \in \Delta([k],[n])$. The functor $S_\bullet : \mathcal{T} \to \mathbf{sSet}$ is called **singular complex**. Given a topological space X in $\mathcal{T}$ it produces a simplicial set $S_\bullet(X)$ whose n-simplices are continuous morphisms $\Delta^n_{\mathcal{T}} \to X$. The jth face morphism $d_j : S_\bullet(X)_n \to S_\bullet(X)_{n-1}$ is given by precomposition with the "inclusion" $|\Delta^{n-1}| \hookrightarrow |\Delta^n|, (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_n)$ so that $d_j : (|\Delta^n| \xrightarrow{f} X) \mapsto (|\Delta^{n-1}| \hookrightarrow |\Delta^n| \xrightarrow{f} X)$. Similarly the jth degeneracy morphism $s_j : S_\bullet(X)_n \to S_\bullet(X)_{n+1}$ is given by precomposition with the map $|\Delta^{n+1}| \to |\Delta^n|, (x_1, \ldots, x_{n+2}) \mapsto (x_1, \ldots, x_{j-1}, x_j + x_{j+1}, x_{j+2}, \ldots, x_{n+2})$. So $s_j : (|\Delta^n| \xrightarrow{f} X) \mapsto (|\Delta^{n+1}| \hookrightarrow |\Delta^n| \xrightarrow{f} X)$. It is straightforward to check that these morphisms satisfy the simplicial relations.

Proposition 4.25. *Geometric realisation and singular complex are an adjoint pair of functors $|\cdot| : \mathbf{sSet} \rightleftarrows \mathcal{T} : S_\bullet$.*

Proof. Let X be an object of **sSet** and let Y be an object of $\mathcal{T}$. Let $f : X \to S_\bullet(Y)$ be a morphism in **sSet**. Given $(t, x) \in |\Delta^n| \times X_n$ define $\hat{f}(t, x) = f(x)(t)$ which is a morphism $|X| \to Y$. Conversely, let $g : |X| \to Y$ be a morphism in $\mathcal{T}$. Given $x \in X_n$ we define $\overline{g}(x)(t) = g(t, x)$ for $t \in |\Delta^n|$. These are clearly inverse constructions which are natural in X and Y. □

The adjunction described above is a Quillen equivalence with respect to the following cofibrantly generated model structure on **sSet**. A proof that this is a model category can be found at Theorem 3.6.5 in [5].

Theorem 4.26. *The category **sSet** of simplicial sets is a cofibrantly generated model structure where the generating cofibrations are the set of boundary inclusions $I = \{\partial\Delta^n \to \Delta^n\}_{n \in \mathbb{N}}$, the generating acyclic cofibrations are the set of horn* *inclusions* $J = \{\Lambda^n_i \to \Delta^n\}_{n\in\mathbb{N},0\leq i\leq n}$, *and the class of weak equivalences* $\mathcal{W}$ *consists of those morphisms* $f : X \to Y$ *whose geometric realisation,* $|f| : |X| \to |Y|$ *is a weak equivalence in* $\mathcal{T}$.

The following lemma is due to Reedy.

Lemma 4.27. *Let* $\mathcal{M}$ *be a model category. If there exists a commutative diagram*

$$\begin{tikzcd}
A \arrow[rr, "b"] \arrow[dr, "a"] \arrow[dd, "s"'] & & B \arrow[dr, "f"] \arrow[dd, "e"', near start] & \\
& A' \arrow[rr, "b'", near start] \arrow[dd, "t", near start] & & B' \arrow[dd, "e'"] \\
C \arrow[rr, "d", near start] \arrow[dr, "c"'] & & D \arrow[dr, "g"] & \\
& C' \arrow[rr, "d'"] & & D'
\end{tikzcd}$$

such that the front and back squares are pushouts, f *is a cofibration, and the morphism* $\phi : A' \amalg_A C \to C'$ *induced by the left square is a cofibration, then* g *is a cofibration.*

Proof. Let $p : X \to Y$ be an acyclic fibration in $\mathcal{M}$. Given a commutative square

$$\begin{tikzcd}
D \arrow[r, "j"] \arrow[d, "g"'] & X \arrow[d, "p"] \\
D' \arrow[r, "k"'] & Y
\end{tikzcd}$$

there exists a lift φ in the following commutative diagram as f is a cofibration and so has the left lifting property with respect to p.

$$\begin{tikzcd}
B \arrow[r, "je"] \arrow[d, "f"'] & X \arrow[d, "p"] \\
B' \arrow[r, "ke'"'] \arrow[ur, dashed, "\varphi"] & Y
\end{tikzcd}$$

Noting that $jds = jeb = \varphi f b = \varphi b' a$, the morphisms $jd : C \to X$ and $\varphi b' : A' \to X$ induce a morphism $\sigma : A' \amalg_A C \to X$. As $\phi : A' \amalg_A C \to C'$ is a cofibration the lift ψ exists in the following commutative diagram.

$$\begin{tikzcd}
A' \amalg_A C \arrow[r, "\sigma"] \arrow[d, "\phi"'] & X \arrow[d, "p"] \\
C' \arrow[r, "kd'"'] \arrow[ur, dashed, "\psi"] & Y
\end{tikzcd}$$

As the front square in the original diagram is a pushout and $\psi t = \psi\phi\iota_{A'} = \varphi b'$ where $\iota_{A'}$ is the pushout of s along a, the morphisms ψ and φ induce a morphism $h : D' \to X$. We have $hge = he'f = \varphi f = je$ and $hgd = hd'c = \psi c = \psi\phi\iota_C = \sigma\iota_C = jd$ where ι_C is the pushout of a along s. As the back square in the original diagram is a pushout this tells that $hg = j$. Further we have that $kd' = p\psi = phd'$ and $ke' = p\varphi = phe'$, as the back square of the original diagram is a pushout this tells us that $ph = k$. Hence the following diagram commutes.

$$\begin{tikzcd} D \arrow[r, "j"] \arrow[d, "g"'] & X \arrow[d, "p"] \\ D' \arrow[r, "k"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

As p was an arbitrary acyclic fibration g has the left lifting property with respect to any acyclic fibration. Hence g is a cofibration. $\square$

Chapter 5

Spectra

In the category of based topological spaces the suspension functor Σ is not invertible. Spectra were introduced to rectify this deficiency. Throughout this chapter $\mathcal{T}_+$ will be the category of compactly generated, weak Hausdorff, based topological spaces with the cofibrantly generated model structure defined in Corollary 4.24. In general, we leave the basepoint of objects in $\mathcal{T}_+$ implicit.

Definition 5.1. The **suspension functor** $\Sigma : \mathcal{T}_+ \to \mathcal{T}_+$ is the functor taking a space X in $\mathcal{T}_+$ to its based suspension $\Sigma X = S^1 \wedge X = (S^1 \times X)/(S^1 \vee X)$. The **loop space functor** $\Omega : \mathcal{T}_+ \to \mathcal{T}_+$ is the functor taking X to $\mathrm{Map}(S^1, X)$ which is the set of basepoint preserving, continuous functions $S^1 \to X$ with the compact-open topology on it.

Proposition 5.2. *The suspension and loop space functors are adjoints.*

Proof. If $f \in \mathcal{T}_+(X, \Omega Y)$ then f is a continuous morphism $X \to \mathrm{Map}(S^1, Y)$. Such a morphism is continuous if and only if (see [10], page 6) the composition $X \times S^1 \xrightarrow{f \times 1_{S^1}} \mathrm{Map}(S^1, Y) \times S^1 \xrightarrow{ev} Y$ where $ev(g, s) = g(s)$ is continuous. As the spaces are based, $f(X \vee S^1) = y_0$ where y_0 is the basepoint in Y. That is, precisely a continuous morphism $(S^1 \times X)/(S^1 \vee X) = \Sigma X \to Y$. This gives a bijection $\mathcal{T}_+(\Sigma X, Y) \cong \mathcal{T}_+(X, \Omega Y)$ which is natural in X and Y. $\square$

Definition 5.3. A **topological category** is a category $\mathcal{D}$ which is enriched over the category of (unbased) compactly generated, weakly Hausdorff topological spaces, $\mathcal{T}$. That is, for any objects d and e in $\mathcal{D}$ the set of morphisms $\mathcal{D}(d, e)$ is an object of $\mathcal{T}$.

Remark 5.4. The above definition is incomplete as we have left out some coherence conditions involving the monoidal structure on $\mathcal{T}$, as we do not make explicit use of them.

Definition 5.5. Given a small topological symmetric monoidal category $\mathcal{D}$ (Defintion 1.22) a **$\mathcal{D}$-space** is an object continuous (limit preserving) functor $\mathcal{D} \to \mathcal{T}_+$. As $\mathcal{D}$ and $\mathcal{T}$ are both enriched over $\mathcal{T}$, for an object X in $\mathcal{DT}$ we have that the induced morphism $\mathrm{Map}(A, B) \to \mathrm{Map}(XA, XB)$ for all $A, B \in \mathrm{obj}(\mathcal{D})$. We write $\mathcal{DT}_+$ for the full subcategory of $\mathcal{T}_+^{\mathcal{D}}$ whose objects are $\mathcal{D}$-spaces. This category is bicomplete with (co)limits constructed level-wise.

We can generalise the adjoint pair $ev_n \dashv (-)_\bullet$ from Definition 2.5. Let $d \in \mathrm{obj}(\mathcal{D})$, then the **$d$th evaluation functor** is $ev_d : \mathcal{DT}_+ \to \mathcal{T}_+$ which sends objects $X : \mathcal{D} \to \mathcal{T}_+$ to X_d and morphisms $f : X \to Y$ in $\mathcal{DT}_+$ which is a natural transformation to the component $f_d : X_d \to Y_d$. The **dth shift desuspension functor** is $F_d : \mathcal{T}_+ \to \mathcal{DT}_+$ which sends objects e to $\mathcal{D}(d, e)_+ \wedge X$ and morphisms $\varphi : e \to f$ in $\mathcal{D}$ to $\mathcal{D}(d, e)_+ \wedge X \xrightarrow{\varphi_* \wedge 1_X} \mathcal{D}(d, f)_+ \wedge X$ where the $_+$ is a disjoint basepoint attached to make it a based space. If $\mathcal{D}(d, e)$ has the discrete topology for all objects d and e in $\mathcal{D}$ then $(F_d X)_e = \mathcal{D}(d, e)_+ \wedge X = \bigvee_{\sigma \in \mathcal{D}(d,e)} X$ which is a copy of X for each map $d \to e$ in $\mathcal{D}$ attached at the basepoint. Then for a morphism $\varphi : e \to f$ we have $F_d(\varphi) : \bigvee_{\sigma \in \mathcal{D}(d,e)} X \to \bigvee_{\sigma \in \mathcal{D}(d,f)} X$ sending the copy of X corresponding to $\sigma \in \mathcal{D}(d, e)$ to the copy of X corresponding to $\varphi\sigma \in \mathcal{D}(d, f)$.

Proposition 5.6. *For all $d \in \mathrm{obj}(\mathcal{D})$ the dth shift desuspension functor and the dth evaluation functor are adjoint, with $F_d \dashv ev_d$.*

Proof. Let $f : F_d X \to Y$ be a morphism in $\mathcal{DT}$, that is f is a natural transformation. So f has components $f_e : (F_d)_e = \mathcal{D}(d, e)_+ \wedge X \to Y_e$. In particular we have $f_d : (F_d X)_d = \mathcal{D}(d, d)_+ \wedge X \to Y_d$. We can restrict this to 1_d to get a morphism $f_d|_{1_{d+} \wedge X} : X \to Y_d = ev_d(Y)$. By the naturality condition on the components of f for any object e in $\mathcal{D}$ and $\varphi \in \mathcal{D}(d, e)$ the following diagram commutes.

$$\begin{tikzcd} (F_d X)_d \arrow[r, "f_d"] \arrow[d, "F_d(\varphi)"'] & Y_d \arrow[d, "Y\varphi"] \\ (F_d X)_e \arrow[r, "f_e"] & Y_e \end{tikzcd}$$

Hence the components of f are entirely determined by $f_d|_{1_{d+} \wedge X}$. Given a morphism $g : X \to Y_d = ev_d(Y)$ in $\mathcal{T}_+$ we let $\tilde{g} : F_d X \to Y$ be the morphism in $\mathcal{DT}$ with components $\tilde{g}_e : (F_d X)_e = \mathcal{D}(d, e)_+ \wedge X \to Y_e, (\sigma, x) \mapsto (Y\sigma)(g(x))$.

□

Proposition 5.7. *Given a small topological symmetric monoidal category $\mathcal{D}$ there is a cofibrantly generated model structure on $\mathcal{DT}$ where the weak equivalences,*

$\mathcal{W}_{\mathcal{DT}}$, are the morphisms $f : X \to Y$ whose component morphisms $f_d : X_d \to Y_d$ are weak equivalences in $\mathcal{T}_+$ for all $d \in \operatorname{obj}(\mathcal{D})$, the set of generating cofibrations is

$$FI = \{F_d(f) \mid f \in I'_+\} = \{F_d(S^n \hookrightarrow D^n)\}_{\substack{d\in\operatorname{obj}(\mathcal{D}),\\ n\in\mathbb{N}}},$$

and the set of generating acyclic cofibrations is

$$FJ = \{F_d(f) \mid f \in J_+\} = \{F_d(D^n \hookrightarrow D^n \times I, x \mapsto (x,0))\}_{\substack{d\in\operatorname{obj}(\mathcal{D}),\\ n\in\mathbb{N}}}.$$

Proof. We prove this by showing that the conditions of Theorem 4.21 are satisfied.

(i) Let f and g be composable morphisms in $\mathcal{DT}$. If two of f, g, and gf are weak equivalences then their components are weak equivalences in $\mathcal{T}_+$. As weak equivalences in $\mathcal{T}_+$ have the two out of three property the component of the third morphism are also weak equivalences, so the third is itself a weak equivalence. Hence the weak equivalences have the two out of three property.

Let $f : A \to B$ be a retract of $g : X \to Y$ where g is a weak equivalence then there exist morphisms in $\mathcal{DT}$ making the first diagram commute. It follows that for each $d \in \operatorname{obj}(\mathcal{D})$ the d components of these morphisms in $\mathcal{T}_+$ make the second diagram commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "1_A"] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "1_B"'] & Y \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} A_d \arrow[r] \arrow[d, "f_d"] \arrow[rr, bend left, "1_{A_d}"] & X_d \arrow[r] \arrow[d, "g_d"] & A_d \arrow[d, "f_d"] \\ B_d \arrow[r] \arrow[rr, bend right, "1_{B_d}"'] & Y_d \arrow[r] & B_d \end{tikzcd}$$

So f_d is a retract of g_d for all d. Then as $\mathcal{T}_+$ is a model category and g_d is a weak equivalence for all d, by the retract axiom, f_d is also a weak equivalence for all $d \in \operatorname{obj}(\mathcal{D})$. So f is a weak equivalence. Hence the weak equivalences are closed under retracts.

(ii) The domains of the FI morphisms are small relative to FI-cell because the domains of the I' morphisms are small relative to I'-cell in $\mathcal{T}_+$

(iii) The domains of the FJ morphisms are small relative to FJ-cell because the domains of the J morphisms are small relative to J-cell in $\mathcal{T}_+$

(iv) Let $f : A \to B$ be a morphism in FJ-cell. Then f is a transfinite composition of pushouts of FJ morphisms. Let the first of the following diagrams be such a pushout. Then by the adjunction $F_{d_i} \dashv ev_{d_i}$ and the fact that colimits are constructed level-wise in $\mathcal{DT}$ the second diagram is also a pushout.

$$\begin{tikzcd} F_{d_i}(D^n) \arrow[r] \arrow[d] & A_i \arrow[d] \\ F_{d_i}(D^n \times I) \arrow[r] & A_{i+1} \end{tikzcd} \qquad \begin{tikzcd} D^n \arrow[r] \arrow[d] & (A_i)_{d_i} \arrow[d] \\ D^n \times I \arrow[r] & (A_{i+1})_{d_i} \end{tikzcd}$$

So $A_i \to A_{i+1}$ is in J-cell in $\mathcal{T}_+$. Then as $\mathcal{T}_+$ is a cofibrantly generated model category $A_i \to A_{i+1}$ is a weak equivalence and is in ${}^{\boxslash}((I')^{\boxslash})$. Hence f is in $\mathcal{W}_{\mathcal{DT}} \cap {}^{\boxslash}(FI^{\boxslash})$.

(v) Let $f : X \to Y$ be a morphism in $FI^{\boxslash}$. Then for all objects d in $\mathcal{D}$ and $n \in \mathbb{N}$ and morphisms j, k such that the outside of the first diagram commutes the dashed lift exists. Then by the adjunction $F_d \dashv ev_d$ the dashed lift also exists in the second diagram.

$$\begin{tikzcd} F_d(S^n) \arrow[r, "j"] \arrow[d] & X \arrow[d, "f"] \\ F_d(D^n) \arrow[r, "k"'] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} S^n \arrow[r, "j_d"] \arrow[d] & X_d \arrow[d, "f_d"] \\ D^n \arrow[r, "k_d"'] \arrow[ur, dashed] & Y_d \end{tikzcd}$$

So f_d is in $(I')^{\boxslash}$ in $\mathcal{T}_+$. Then as $\mathcal{T}_+$ is a cofibrantly generated model category f_d is a weak equivalence and is in $J^{\boxslash}$. Hence f is in $\mathcal{W}_{\mathcal{DT}} \cap FJ^{\boxslash}$.

(vi) Let $p : X \to Y$ be a morphism in $\mathcal{W}_{\mathcal{DT}} \cap FJ^{\boxslash}$. Then for all $n \in \mathbb{N}$, objects d in $\mathcal{D}$, and morphisms j and k such that the outside of the first diagram commutes, the dashed lift exists. Then by the adjunction $F_d \dashv ev_d$ the dashed lift also exists in the second diagram.

$$\begin{tikzcd} F_d(D^n) \arrow[r, "j"] \arrow[d] & X \arrow[d, "p"] \\ F_d(D^n \times I) \arrow[r, "k"'] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} D^n \arrow[r, "j_d"] \arrow[d] & X_d \arrow[d, "p_d"] \\ D^n \times I \arrow[r, "k_d"'] \arrow[ur, dashed] & Y_d \end{tikzcd}$$

Hence p_d is in $J^{\boxslash}$. As p is a weak equivalence, so is p_d. Hence as $\mathcal{T}_+$ is a cofibrantly generated model category p_d is in $(I')^{\boxslash}$. Hence p is in $FI^{\boxslash}$.

$\square$

Recall that $\mathcal{T}_+$ is a closed, symmetric monoidal category (Example 1.24). We would like to use the monoidal product on $\mathcal{T}_+$ (the smash product) to define a monoidal product on $\mathcal{D}$-spaces. The obvious thing to do would be to define the level-wise smash product $X \bar{\wedge} Y$ of two $\mathcal{D}$-spaces by $(X \bar{\wedge} Y)_e = X_e \wedge Y_e$. Unfortunately $(X \bar{\wedge} Y) : \mathcal{D} \times \mathcal{D} \to \mathcal{T}_+$ may not be a $\mathcal{D}$-space. Instead let $\odot$ be the monoidal product in $\mathcal{D}$ and let $X \wedge Y$ be the left Kan extension of $X \bar{\wedge} Y$ along $\odot$. This gives us the following definition.

Definition 5.8. Let $\mathcal{D}$ be a small symmetric monoidal category with monoidal product $\odot$. Let X and Y be $\mathcal{D}$-spaces, then the **internal smash product** $X \wedge Y$ is a $\mathcal{D}$-space with $(X \wedge Y)_d = \operatorname*{colim}_{(\sigma:e\odot f\to d)\in(\odot\downarrow d)} (X_e \wedge Y_f)$. Alternatively, this can be written as the coend

$$(X \wedge Y)_d = \int^{(e,f)\in\mathcal{D}\times\mathcal{D}} \mathcal{D}(e\odot f, d)_+ \wedge (X_e \wedge Y_f)$$

$$= \operatorname{coeq}\left(\coprod_{\substack{(\sigma:(e,f)\to(e',f'))\\ \in\mathcal{D}\times\mathcal{D}}} (\mathcal{D}(e\odot f,d)_+ \wedge (X_{e'} \wedge Y_{f'})) \rightrightarrows \coprod_{(e,f)\in\mathrm{obj}(\mathcal{D}\times\mathcal{D})} (\mathcal{D}(e\odot f,d)_+ \wedge (X_e \wedge Y_f)) \right).$$

If $\mathcal{D}$ is a topological category where the morphism sets between objects are given the discrete topology this becomes

$$(X \wedge Y)_d =$$

$$\operatorname{coeq}\left(\bigvee_{\substack{(\sigma:(e,f)\to(e',f'))\\ \in\mathcal{D}\times\mathcal{D}}} \left(\bigvee_{\psi\in\mathcal{D}(e\odot f,d)} (X_{e'} \wedge Y_{f'}) \right) \rightrightarrows \bigvee_{(e,f)\in\mathrm{obj}(\mathcal{D}\times\mathcal{D})} \left(\bigvee_{\psi\in\mathcal{D}(e\odot f,d)} (X_e \wedge Y_f) \right) \right).$$

That is, the coequaliser of the morphisms induced by the following families of morphisms.

$$\left\{ \bigvee_{\psi\in\mathcal{D}(e\odot f,d)} (X_{e'} \wedge Y_{f'}) \xrightarrow{\omega} \bigvee_{\psi\in\mathcal{D}(e'\odot f',d)} (X_{e'} \wedge Y_{f'}) \xrightarrow{\chi_{(e',f')}} \bigvee_{(e'',f'')\in\mathrm{obj}(\mathcal{D}\times\mathcal{D})} \left(\bigvee_{\psi\in\mathcal{D}(e''\odot f'',d)} (X_{e''} \wedge Y_{f''}) \right) \right\}_{(\sigma:(e',f')\to(e,f))\in\mathcal{D}\times\mathcal{D}}$$

and

$$\left\{ \bigvee_{\psi\in\mathcal{D}(e\odot f,d)} (X_{e'} \wedge Y_{f'}) \xrightarrow{\xi} \bigvee_{\psi\in\mathcal{D}(e\odot f,d)} (X_e \wedge Y_f) \xrightarrow{\chi_{(e,f)}} \bigvee_{(e'',f'')\in\mathrm{obj}(\mathcal{D}\times\mathcal{D})} \left(\bigvee_{\psi\in\mathcal{D}(e''\odot f'',d)} (X_{e''} \wedge Y_{f''}) \right) \right\}_{(\sigma:(e',f')\to(e,f))\in\mathcal{D}\times\mathcal{D}}$$

Here ω is the morphism sending the copy of $X_{e'} \wedge Y_{f'}$ corresponding to $\psi : e\odot f \to d$ to the copy of $X_{e'} \wedge Y_{f'}$ corresponding to $\psi\sigma : e' \odot f' \to d$ (along the identity) and, ξ sends the copy of $X_{e'} \wedge Y_{f'}$ corresponding to ψ to the copy of $X_e \wedge Y_f$ corresponding to ψ along the map $X\sigma_1 \wedge Y\sigma_2$ where $\sigma = (\sigma_1 : e' \to e, \sigma_2 : f' \to f)$. The morphisms $\chi_{(e',f')}$ and $\chi_{(e,f)}$ are the colimit injection morphisms.

With this internal smash product $\mathcal{DT}$ is a closed symmetric monoidal category. See [7] page 506 for a full proof of the following theorem and for the definition of the internal hom.

Theorem 5.9. *Let $\mathcal{D}$ be a small symmetric monoidal category with identity object u. Then the category of $\mathcal{D}$-spaces is a closed symmetric monoidal category where the monoidal product is the internal smash product defined above, and the identity object is $F_u S^0$.*

Definition 5.10. A **$\mathcal{D}$-spectrum** is a $\mathcal{D}$-space which is an $\mathbb{S}$-module where $\mathbb{S}$ is the "obvious" sphere functor for the choice of $\mathcal{D}$. That is, a $\mathcal{D}$-space X together with a morphism $\mathbb{S} \wedge X \to X$. We denote the category of $\mathcal{D}$-spectra by $\mathrm{Sp}^{\mathcal{D}}$.

Remark 5.11. This can be generalised to $\mathcal{D}$-spectra over R where R is any monoid in $\mathcal{DT}$. See [7] page 449 for details.

We will now describe some important examples of $\mathcal{D}$-spectra.

Definition 5.12. Let $\mathcal{N}$ be the category whose objects are the natural numbers, $\mathbb{N}$, and whose only morphisms are identity morphisms. This is a small topological symmetric monoidal category where the monoidal product is addition, and the identity object is 0. The morphism spaces are all either the empty space or the one point space. The sphere functor in this case is $\mathbb{S} : \mathcal{N} \to \mathcal{T}_+, n \mapsto S^n$. We call $\mathcal{N}$-spectra **prespectra**. For an $\mathcal{N}$-space X to be a prespectrum we require a morphism $\alpha : \mathbb{S} \wedge X \to X$. First consider the domain of α_n, as there are only identity morphisms in $\mathcal{N}$ for $n \in \mathbb{N}$ we have

$$(\mathbb{S} \wedge X)_n = \operatorname*{colim}_{\sigma : e+f \to n \in (+\downarrow n)} \mathbb{S}_e \wedge X_f = (S^n \wedge X_0) \vee (S^{n-1} \wedge X_1) \vee \ldots \vee (S^1 \wedge X_{n-1}) \vee (S^0 \wedge X_n).$$

Hence the nth component of α is given by morphisms $S^p \wedge X_{n-p} \to X_n$ for all $0 \leq p \leq n$. Thus, equivalently a prespectrum is a sequence of spaces $\{X_n\}_{n \in \mathbb{N}}$ in $\mathcal{T}_+$ together with morphisms $S^m \wedge X_n \to X_{m+n}$ for all $n, m \in \mathbb{N}$.

Definition 5.13. Let Σ be the category whose objects are the sets $[n] = \{1, \ldots, n\}$ for all $n \in \mathbb{N}$ and whose morphisms are permutations of $[n]$ for each $n \in \mathbb{N}$, that is $\Sigma([n], [n])$ is the symmetric group on n letters, Σ_n. This also means there are only morphisms $[n] \to [m]$ in Σ if $n = m$. This is a small topological symmetric monoidal category where the monoidal product is addition, that is $[n] + [m] = [n + m]$, and the identity object is $[0]$ which is the empty set. The morphism sets are given the discrete topology. The sphere functor in this case is

$\mathbb{S} : \mathcal{N} \to \mathcal{T}_+, n \mapsto S^n \cong S^1 \wedge \ldots \wedge S^1$ where the image of $\sigma \in \Sigma([n],[n])$ under $\mathbb{S}$ permutes the smash factors of S^n. We call Σ-spectra **symmetric spectra**. For a σ-space to be a symmetric spectrum we require the a morphism $\alpha : \mathbb{S} \wedge X \to X$. First consider the domain of α_n, we have

$$(\mathbb{S} \wedge X)_n = \int^{(i,j)\in\Sigma\times\Sigma} \mathcal{D}(i+j,n)_+ \wedge (S^i \wedge X_j)$$

$$= \operatorname{coeq}\left(\bigvee_{\substack{(\sigma:(i,j)\to(i',j'))\\ \in\Sigma\times\Sigma}} \left(\bigvee_{\psi\in\mathcal{D}(i+j,n)} (S^{i'} \wedge X_{j'}) \right) \rightrightarrows \bigvee_{(i,j)\in\mathrm{obj}(\Sigma\times\Sigma)} \left(\bigvee_{\psi\in\mathcal{D}(i+j,n)} (S^i \wedge X_j) \right) \right).$$

Since there are only morphisms $i + j \to n$ if $i + j = n$, and morphisms $i' \to i$ and $j' \to j$ if $i' = i$ and $j' = j$ this is the coequaliser of the parallel morphisms induced by the following families of morphisms.

$$\left\{ \bigvee_{\psi\in\Sigma_n}(S^i \wedge X_{n-i}) \xrightarrow{\omega} \bigvee_{\psi\in\Sigma_n}(S^i \wedge X_{n-i}) \to \bigvee_{i'=0}^{n} \left(\bigvee_{\psi\in\Sigma_n} (S^{i'} \wedge X_{n-i'}) \right) \right\}_{\substack{(\sigma_1,\sigma_2)\in\Sigma_i\times\Sigma_{n-i}\\ 0\le i\le n}}$$

and

$$\left\{ \bigvee_{\psi\in\Sigma_n}(S^i \wedge X_{n-i}) \xrightarrow{\xi} \bigvee_{\psi\in\Sigma_n}(S^i \wedge X_{n-i}) \to \bigvee_{i'=0}^{n} \left(\bigvee_{\psi\in\Sigma_n} (S^{i'} \wedge X_{n-i'}) \right) \right\}_{\substack{(\sigma_1,\sigma_2)\in\Sigma_i\times\Sigma_{n-i}\\ 0\le i\le n}}$$

Here ω takes the copy of $S^i \wedge X_{n-i}$ corresponding the $\phi : [n] \to [n]$ to the copy corresponding to $\phi(\sigma_1 + \sigma_2)$ and the second takes the copy of $S^i \wedge X_{n-i}$ corresponding to ϕ to the same copy by the morphism $S^i \wedge X_{n-i} \xrightarrow{\mathbb{S}\sigma_1 \wedge X\sigma_2} S^i \wedge X_{n-i}$. In the first case we are implicitly thinking about $\Sigma_i \times \Sigma_{n-i}$ as a subgroup of Σ_n where the first component acts on the first i letters and the second acts on the last $n - i$ letters. So a morphism $(\mathbb{S} \wedge X)_n$ is equivalent to a $(\Sigma_i \wedge \Sigma_{n-i})$-equivariant morphism $S^i \wedge X_{n-i} \to X_n$ for each $0 \le i \le n$. So a symmetric spectra is a sequence of spaces $\{X_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}_+$ together with $(\Sigma_i \wedge \Sigma_{n-i})$-equivariant morphisms $S^i \wedge X_{n-i} \to X_n$ for all $n \in \mathbb{N}$ and $0 \le i \le n$.

Definition 5.14. Let $\mathscr{I}$ be the category whose objects are finite dimensional real inner product spaces and whose morphisms are linear isometric isomorphisms. This means there are only morphisms $V \to W$ if $\dim(V) = \dim(W) = n$ for some $n \in \mathbb{N}$. When this is the case, $\mathscr{I}(V, W)$ is homeomorphic to the orthogonal group O_n. This is a small topological symmetric monoidal category where the monoidal product is direct sum, and the identity object is the 0-dimensional space. The sphere functor in this case is $\mathbb{S} : \mathscr{I} \to \mathcal{T}_+, V \mapsto S^V$ where S^V is the 1 point compactification of V where $\mathbb{S}$ takes a morphism $V \to W$ to the obvious morphism $S^V \to S^W$. We call $\mathscr{I}$-spectra **orthogonal spectra**.

Definition 5.15. Restricting the evaluation functors from Definition 5.5 to $\mathcal{D}$-spectra gives a functor $ev_d : \mathrm{Sp}^{\mathcal{D}} \to \mathcal{T}_+, X \to X_d$ this is the d**th evaluation functor** on $\mathcal{D}$-spectra. These evaluation functors have left adjoints $F_d : \mathcal{T}_+ \to \mathrm{Sp}^{\mathcal{D}}$ which we will call the d**th shift desuspension functor** to $\mathcal{D}$-spectra. We will describe these functors for $\mathcal{D} = \mathcal{N}$ and $\mathcal{D} = \Sigma$.

For $\mathcal{D} = \mathcal{N}$ the ith evaluation functor $F_i : \mathcal{T}_+ \to \mathrm{Sp}^{\mathcal{N}}$ takes a space X in $\mathcal{T}_+$ to the presectra with

$$(F_i X)_j = \begin{cases} S^{j-i} \wedge X & \text{for } j \geq i \\ \{*\} & \text{for } j < i \end{cases}$$

For $\mathcal{D} = \Sigma$ the ith evaluation functor $F_i : \mathcal{T}_+ \to \mathrm{Sp}^{\mathcal{N}}$ takes a space X in $\mathcal{T}_+$ to the symmetric spectra with

$$(F_i X)_j = \begin{cases} \Sigma_{j_+} \wedge_{\Sigma_{j-i}} (S^{j-i} \wedge X) & \text{for } j \geq i \\ \{*\} & \text{for } j < i \end{cases}$$

Theorem 5.16. *Let $\mathcal{D}$ be one of $\mathcal{N}$, Σ, or $\mathscr{I}$. There is a* ***level model structure*** *on the category of $\mathcal{D}$-spectra, where $f : X \to Y$ is a level weak equivalence (fibration) if it is a level weak equivalence (fibration) of $\mathcal{D}$-spaces, and f is a q-cofibration if it has the left lifting property with respect to all level acyclic fibrations.*

Freudanthal's suspension theorem (see [2] chapter 4 section 2) states that if X is an $(n-1)$-connected CW complex then the suspension map on homotopy groups $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is a surjection for $i = 2n-1$ and is an isomorphism for $i < 2n-1$. It follows that for each $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\operatorname{colim}_n \pi_{q+n}(\Sigma^n X) \cong \pi_{q+n}(\Sigma^n X)$. This leads to the study of stable homotopy theory. To study stable homotopy we want the weak equivalences to include those morphisms which induce an isomorphism on stable homotopy groups. Unfortunately there are morphisms which do this which are not weak equivalences in our level model structure. To rectify this we add more weak equivalences, a process called localisation. However, changing just one of the classes of morphisms in a model structure will prevent the lifting axiom from being satisfied. So we fix one of the other classes and change the third to be the class determined by the fixed class and the new class of weak equivalences. In the case of spectra we fix the cofibrations. As there are acyclic cofibrations the new class of fibrations is contained in the previous class.

Definition 5.17. The q**th homotopy group** of a prespectrum X is defined to be

$$\pi_q(X) = \operatorname*{colim}_n(\pi_{q+n}(X_n)).$$

Where $\pi_i(X_n)$ is the ith homotopy group of X_n as a based topological space. A morphism of prespectra $f : X \to Y$ is a natural transformation with components $f_j : X_j \to Y_j$ for all $j \in \mathbb{N}$ which, for each $k \in \mathbb{N}$, induces a group homomorphism $\pi_k(X_j) \to \pi_k(Y_j)$. Hence f induces a group homomorphism $\pi_n(X) \to \pi_n(Y)$ for all $n \in \mathbb{N}$. We say that a morphism of prespectra is a π_***-isomorphism** if it induces an isomorphism $\pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ for all $n \in \mathbb{N}$.

Definition 5.18. Let X be a prespectrum with structure morphisms $\sigma_n : S^1 \wedge X_n = \Sigma X \to X_{n+1}$ for all $n \in \mathbb{N}$. As the suspension and loop space functors are adjoint (Proposition 5.2) these morphisms have adjoints $\tilde{\sigma_n} : X_n \to \Omega X_{n+1}$. We call X an Ω**-spectrum** if $\tilde{\sigma_n}$ is a weak equivalence in $\mathcal{T}_+$ for all $n \in \mathbb{N}$.

Definition 5.19. There are inclusion functors between our $\mathcal{D}$ categories, $\mathcal{N} \to \Sigma, n \mapsto n$ and $\Sigma \to \mathscr{I}, n \mapsto \mathbb{R}^n$. Their composition $\mathcal{N} \to \mathscr{I}$ is also an inclusion functor. For $\iota : \mathcal{A} \to \mathcal{B}$ such an inclusion we define the **forgetful functor** to be $\mathbb{U} : \mathcal{B}\mathcal{T} \to \mathcal{A}\mathcal{T}$ with $(\mathbb{U}X)_a = X(\iota(a))$ for $a \in \operatorname{obj}(\mathcal{A})$. These forgetful functors have right adjoints $\mathbb{P} : \mathcal{A}\mathcal{T} \to \mathcal{B}\mathcal{T}$ called **prolongation functors**. For X an $\mathcal{A}$-space $\mathbb{P}X$ is the left Kan extension of X along ι. In particular, for $a \in \operatorname{obj}(\mathcal{A})$ we have

$$(\mathbb{P}X)_b = \operatorname*{colim}_{(\iota a \to b) \in (\iota \downarrow b)} X_a = \int^{a \in \mathcal{A}} \mathcal{B}(\iota a, b) \wedge X_a$$

Definition 5.20. Let $\mathcal{D}$ be either Σ or $\mathscr{I}$ and let $[X, Y]$ denote the set of morphisms $X \to Y$ in the homotopy category $\operatorname{Ho}\mathcal{D}$ for any $\mathcal{D}$-spectra X and Y. We define the following:

(i) A $\mathcal{D}$-spectra E is a $\mathcal{D}$**-Ω-spectrum** if $\mathbb{U}E$ is an Ω-specturm.

(ii) A morphism of $\mathcal{D}$-spectra $f : X \to Y$ is a π_***-isomorphism** if $\mathbb{U}f$ is a π_*-isomorphism.

(iii) A morphism of $\mathcal{D}$-spectra $f : X \to Y$ is a **stable equivalence** if $f^* : [Y, E] \to [X, E]$ is a bijection for all $\mathcal{D}$-Ω-spectra, E.

For details of the proof of the following theorem see page 471 of [7].

Theorem 5.21. *Let $\mathcal{D}$ be one of $\mathscr{N}$, Σ, or $\mathscr{I}$. There is a* ***stable model structure*** *on the category of $\mathcal{D}$-spectra, where the weak equivalences are the stable equivalences, the cofibrations are the q-cofibrations, and the fibrations (called q-fibrations) are morphisms with the right lifting property with respect to the acyclic q-cofibrations.*

Theorem 5.22. *Let $\mathbb{U} \dashv \mathbb{P}$ be an adjoint pair of forgetful and prolongation functors as defined above. Then $(\mathbb{U}, \mathbb{P}, \varphi)$ is a Quillen equivalence.*

This theorem shows that the homotopy categories of the $\mathcal{D}$-spectra are equivalent for the choices of $\mathcal{D}$ we have described above. This is an important result because in [6] the author describes a collection of axioms that one might want a category of spectra to satisfy then demonstrates that it is not possible to satisfy all of them at the same time. The existence of Quillen equivalences between the different categories of spectra allows one to choose the most convenient for the current task.

Chapter 6

Reedy Model Categories

In [4] Hirschhorn and Volić characterise the Reedy functors $\mathcal{C} \to \mathcal{D}$ for which the induced functor $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a left or right Quillen functor (for the Reedy model structure) for all model categories $\mathcal{M}$. Their paper characterises the case for right Quillen functors and then establishes the characterisation for left Quillen functors by discussing opposite categories. Here we establish the characterisation for left Quillen functors by dualising the the argument presented in [4] for right Quillen functors.
A Reedy category is a category that admits an assignation of degree to each object and a unique factorisation of any morphism into a morphism decreasing degree followed by a morphism increasing degree.

Definition 6.1. A **Reedy category** is a small category $\mathcal{C}$ in which each object can be assigned a non-negative integer (its **degree**) together with subcategories $\overleftarrow{\mathcal{C}}$ and $\overrightarrow{\mathcal{C}}$, each containing all the objects of $\mathcal{C}$ and satisfying the following axioms;

1. The target of every non-identity map in $\overleftarrow{\mathcal{C}}$ is of strictly lower degree than the source.
2. The target of every non-identity map in $\overrightarrow{\mathcal{C}}$ is of strictly higher degree than the source.
3. Every map g in $\mathcal{C}$ has a unique factorisation $g = \overrightarrow{g}\overleftarrow{g}$ where $\overrightarrow{g}$ is in $\overrightarrow{\mathcal{C}}$ and $\overleftarrow{g}$ is in $\overleftarrow{\mathcal{C}}$.

The subcategories $\overleftarrow{\mathcal{C}}$ and $\overrightarrow{\mathcal{C}}$ are called the inverse subcategory and direct subcategory respectively.

Examples 6.2. (i) Let X be a finite poset. We can consider the category $\mathcal{X}$ whose objects are elements of X such that $\mathcal{X}(x, y) = \{*\}$ if $x \preceq y$ and

$\mathcal{X}(x, y) = \varnothing$ otherwise. If $x \in X$ is a minimal element let $\deg(x) = 0$. Otherwise there is some set of minimal elements $M_x = \{m \in X \,|\, \mathcal{X}(m, x) = \{*\}\}$, let $\deg(x)$ be the largest $n \in \mathbb{N}$ such that there is a sequence $m \xrightarrow{f_1} a_1 \to \ldots \to a_{n-1} \xrightarrow{f_n} x$ in $\mathcal{X}$ for some $m \in M_x$. Then $\mathcal{X}$ is a Reedy category where $\overleftarrow{\mathcal{X}}$ includes only identity morphisms, $\overrightarrow{\mathcal{X}} = \mathcal{X}$, and the factorisation is given by $\overrightarrow{f} = f$ and $\overleftarrow{f}$ is the identity of the source of f. This naturally extends to any countable poset with a unique minimal element such as $\mathbb{N}$.

(ii) Reedy categories are a generalisation of the Δ category (Definition 2.1). In particular, Δ is a Reedy category with $\deg([n]) = n$, the direct subcategory contains the injective morphisms, the inverse subcategory contains the surjective morphisms, and the factorisation of a morphism is the one described in Proposition 2.3.

(iii) The category consisting of a parallel pair of morphisms $\cdot \rightrightarrows \cdot$ is a Reedy category where both non-identity arrows strictly raise degree.

(iv) Given a Reedy category $\mathcal{R}$, the opposite category $\mathcal{R}^{\mathrm{op}}$ is a Reedy category with the same degree assignation on objects, $\overrightarrow{\mathcal{R}^{\mathrm{op}}} = (\overleftarrow{\mathcal{R}})^{\mathrm{op}}$, $\overleftarrow{\mathcal{R}^{\mathrm{op}}} = (\overrightarrow{\mathcal{R}})^{\mathrm{op}}$, and the factorisation given by $f^{\mathrm{op}} = (\overleftarrow{f})^{\mathrm{op}}(\overrightarrow{f})^{\mathrm{op}}$.

(v) Given two Reedy categories $\mathcal{R}$ and $\mathcal{P}$, the product category $\mathcal{R} \times \mathcal{P}$ is a Reedy category with $\deg(r, p) = \deg(r) + \deg(p)$, $\overrightarrow{\mathcal{R} \times \mathcal{P}} = \overrightarrow{\mathcal{R}} \times \overrightarrow{\mathcal{P}}$, $\overleftarrow{\mathcal{R} \times \mathcal{P}} = \overleftarrow{\mathcal{R}} \times \overleftarrow{\mathcal{P}}$, and the factorisation of (f, g) given by $\overrightarrow{(f, g)} = (\overrightarrow{f}, \overrightarrow{g})$ and $\overleftarrow{(f, g)} = (\overleftarrow{f}, \overleftarrow{g})$.

Given two Reedy categories we can consider functors between them which preserve the Reedy structure.

Definition 6.3. Let $\mathcal{C}, \mathcal{D}$ be Reedy categories. A **Reedy functor** is a functor $G : \mathcal{C} \to \mathcal{D}$ such that $F(\overleftarrow{\mathcal{C}})$ is a subcategory of $\overleftarrow{\mathcal{D}}$ and $F(\overrightarrow{\mathcal{C}})$ is a subcategory of $\overrightarrow{\mathcal{D}}$. That is, objects and morphisms in the inverse and direct subcategories of $\mathcal{C}$ are sent to objects and morphisms in the inverse and direct subcategories of $\mathcal{D}$ respectively.

Definition 6.4. Let $\mathcal{C}$ be a Reedy category, $\mathcal{M}$ be a model category, $f : X \to Y$ be a natural transformation between functors $X, Y : \mathcal{C} \to \mathcal{M}$, and $\alpha \in \mathrm{obj}\,(\mathcal{C})$.

(i) The **latching category**, $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ of $\mathcal{C}$ at α is the full subcategory of $(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ containing all the objects except the identity map at α. Recalling

Definition 1.2 and Notation 1.4, the objects of the latching category of $\mathcal{C}$ at α are pairs (β, f) where β is an object of $\overrightarrow{\mathcal{C}}$ and $f : \beta \to \alpha$ is a morphism in $\overrightarrow{\mathcal{C}}$, and the morphisms $(\beta, f) \to (\beta', f')$ are morphisms $\tau : \beta \to \beta'$ in $\overrightarrow{\mathcal{C}}$ such that the following diagram commutes.

$$\begin{tikzcd}
\beta \arrow[rr, "\tau"] \arrow[dr, "f"'] & & \beta' \arrow[dl, "f'"] \\
& \alpha &
\end{tikzcd}$$

(ii) The **latching object** of X at α is

$$L_\alpha X = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)} X$$

As written this is not well defined because $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ is not a subcategory of $\mathcal{C}$. By an abuse of notation we will identify $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ with the subcategory of $\mathcal{C}$ in the image of the functor $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \to \mathcal{C}$ taking objects (β, f) to β and morphisms $\tau : \beta \to \beta'$ to themselves.

(iii) The **latching map** of X at α is the natural morphism

$$L_\alpha X \to X_\alpha$$

(iv) The **relative latching map** of $f : X \to Y$ at α is the natural morphism

$$X_\alpha \amalg_{L_\alpha X} L_\alpha Y \to Y_\alpha$$

(v) The **matching category**, $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ of $\mathcal{C}$ at α is the full subcategory of $(\alpha \downarrow \overleftarrow{\mathcal{C}})$ containing all the objects except the identity map at α.

(vi) The **matching object** of X at α is

$$M_\alpha X = \lim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})} X$$

(vii) The **matching map** of X at α is the natural morphism

$$X_\alpha \to M_\alpha X$$

(viii) The **relative matching map** of $f : X \to Y$ at α is the natural morphism

$$X_\alpha \to M_\alpha X \times_{M_\alpha Y} Y_\alpha$$

If $\mathcal{C}$ is the category Δ^{op} then for a Δ^{op}-diagram, X, in $\mathcal{M}^{\Delta^{\mathrm{op}}}$ (that is for a simplicial object in $\mathcal{M}$) the latching object at $[n] \in \mathrm{obj}\,(\Delta^{\mathrm{op}})$ is $L_\alpha X = (\mathrm{sk}_{n-1}X)_n$.

Given a Reedy category $\mathcal{C}$ and a model category $\mathcal{M}$ the functor category $\mathcal{M}^{\mathcal{C}}$ is a model category.

Theorem 6.5. *The diagram category $\mathcal{M}^{\mathcal{C}}$ has a model structure on it, where a morphism $f : X \to Y$ is*

- *a weak equivalence if for all $\alpha \in obj(\mathcal{C})$ the component $f_\alpha : X_\alpha \to Y_\alpha$ is a weak equivalence in $\mathcal{M}$*
- *a cofibration if for all $\alpha \in obj(\mathcal{C})$ the relative latching map $X_\alpha \amalg_{L_\alpha X} L_\alpha Y \to Y_\alpha$ is a cofibration in $\mathcal{M}$*
- *a fibration if for all $\alpha \in obj(\mathcal{C})$ the relative matching map $X_\alpha \to M_\alpha X \times_{M_\alpha Y} Y_\alpha$ is a fibration in $\mathcal{M}$.*

Proof. This result is due to Daniel Kan. See [3], Theorem 15.3.4. □

Given a Reedy functor $G : \mathcal{C} \to \mathcal{D}$ we can consider factorisations of morphisms in $\overleftarrow{\mathcal{D}}$ whose source is in the image of G where the first morphism is in $G(\overleftarrow{\mathcal{C}})$.

Definition 6.6. Let $G : \mathcal{C} \to \mathcal{D}$ be a Reedy functor, $\alpha \in \mathrm{obj}\,(\mathcal{C})$, $\beta \in \mathrm{obj}\,(\mathcal{D})$, and $\sigma : G\alpha \to \beta$ be in $\overleftarrow{\mathcal{D}}$. The **category of inverse $\mathcal{C}$-factorizations** of (α, σ), denoted $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, \sigma)$, is the category where

- objects are pairs $((\nu : \alpha \to \gamma), (\mu : G\gamma \to \beta))$ where ν is a non-identity map in $\overleftarrow{\mathcal{C}}$ and μ is in $\overleftarrow{\mathcal{D}}$ such that the following diagram commutes

$$\begin{tikzcd} G\alpha \arrow[rr, "G\nu"] \arrow[dr, "\sigma"'] & & G\gamma \arrow[dl, "\mu"] \\ & \beta & \end{tikzcd}.$$

- morphisms between pairs

$$((\nu : \alpha \to \gamma), (\mu : G\gamma \to \beta)) \to ((\nu' : \alpha \to \gamma'), (\mu' : G\gamma' \to \beta))$$

  are maps $\tau : \gamma \to \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that the following diagrams commute

$$\begin{tikzcd} & \alpha \arrow[dl, "\nu"'] \arrow[dr, "\nu'"] & \\ \gamma \arrow[rr, "\tau"] & & \gamma' \end{tikzcd} \qquad \begin{tikzcd} G\gamma \arrow[rr, "G\tau"] \arrow[dr, "\mu"'] & & G\gamma' \arrow[dl, "\mu'"] \\ & \beta & \end{tikzcd}.$$

Similarly we can define the category of direct $\mathcal{C}$-factorizations.

Definition 6.7. Let $G : \mathcal{C} \to \mathcal{D}$ be a Reedy functor, $\alpha \in \mathrm{obj}(\mathcal{C})$, $\beta \in \mathrm{obj}(\mathcal{D})$, and $\sigma : \beta \to G\alpha$ be in $\overrightarrow{\mathcal{D}}$. The **category of direct $\mathcal{C}$-factorizations** of (α, σ), denoted $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$, is the category where

- objects are pairs $((\nu : \gamma \to \alpha), (\mu : \beta \to G\gamma))$ where ν is a non-identity map in $\overrightarrow{\mathcal{C}}$ and μ is in $\overrightarrow{\mathcal{D}}$ such that the following diagram commutes

$$\begin{tikzcd} \beta \arrow[rr, "\mu"] \arrow[dr, "\sigma"'] & & G\gamma \arrow[dl, "G\nu"] \\ & G\alpha & \end{tikzcd}.$$

- morphisms between pairs

$$((\nu : \gamma \to \alpha), (\mu : \beta \to G\gamma)) \to ((\nu' : \gamma' \to \alpha), (\mu' : \beta \to G\gamma'))$$

are maps $\tau : \gamma \to \gamma'$ in $\overrightarrow{\mathcal{C}}$ such that the following diagrams commute

$$\begin{tikzcd} & \alpha & \\ \gamma \arrow[ur, "\nu"] \arrow[rr, "\tau"] & & \gamma' \arrow[ul, "\nu'"'] \end{tikzcd} \qquad \begin{tikzcd} G\gamma \arrow[rr, "G\tau"] & & G\gamma' \\ & \beta \arrow[ul, "\mu"] \arrow[ur, "\mu'"'] & \end{tikzcd}.$$

Remark 6.8. A Reedy functor induces a functor for each object in the source category from its latching category to the under category of its image in the direct subcategory. That is, let $G : \mathcal{C} \to \mathcal{D}$ be a Reedy functor, and let $\alpha \in \mathrm{obj}(\mathcal{C})$ then G induces a functor $G_* : \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \to (\overrightarrow{\mathcal{D}} \downarrow G\alpha)$ which takes $\gamma \to \alpha$ to $G\gamma \to G\alpha$.

Proposition 6.9. *Let G and α be as above, let $\beta \in \mathrm{obj}(\mathcal{D})$, and let $\sigma : \beta \to G\alpha$ be a morphism in $\overrightarrow{\mathcal{D}}$. Then the category of direct $\mathcal{C}$-factorisations, $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$ is the category of the image of objects of the latching category under σ, that is, $(\sigma \downarrow G_*)$.*

Proof. An object in $(\sigma \downarrow G_*)$ is a pair (ν, μ) where ν is an object of $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$, that is a non-identity morphism $\nu : \gamma \to \alpha$ in $\overrightarrow{\mathcal{C}}$ and μ is a morphism $\sigma \to G\nu$ in $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$, that is a morphism $\mu : \beta \to G\gamma$ in $\overrightarrow{\mathcal{D}}$ such that the following diagram commutes.

$$\begin{tikzcd} \beta \arrow[rr, "\mu"] \arrow[dr, "\sigma"'] & & G\gamma \arrow[dl, "G\nu"] \\ & G\alpha & \end{tikzcd}$$

A morphism $(\nu, \mu) \to (\nu', \mu')$ in $(\sigma \downarrow G_*)$ is a morphism $\tau : \nu \to \nu'$ in $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ such that the following diagram commutes.

$$\begin{tikzcd} & (\beta, \sigma) \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] & \\ G_*(\gamma, \nu) \arrow[rr, "G\tau"'] & & G_*(\gamma', \nu') \end{tikzcd}$$

That is, a non-identity morphism $\tau : \gamma \to \gamma'$ in $\overrightarrow{\mathcal{C}}$ such that the following diagrams commute.

$$\begin{tikzcd} & \alpha & \\ \gamma \arrow[ur, "\nu"] \arrow[rr, "\tau"] & & \gamma' \arrow[ul, "\nu'"'] \end{tikzcd} \qquad \begin{tikzcd} G\gamma \arrow[rr, "G\tau"] & & G\gamma' \\ & \beta \arrow[ul, "\mu"] \arrow[ur, "\mu'"'] & \end{tikzcd}.$$

These are precisely the objects and morphisms of $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$. □

Remark 6.10. In the case where G takes non-identity morphisms to non-identity morphisms the induced functor G_* is from the latching category of α to the latching category of $G\alpha$.

Now we define the class of Reedy functors used in the classification.

Definition 6.11. Let $G : \mathcal{C} \to \mathcal{D}$ be a Reedy functor. It is a **cofibring Reedy functor** if for every $\alpha \in \mathrm{obj}(\mathcal{C})$, for every $\beta \in \mathrm{obj}(\mathcal{D})$, and for every map $\sigma : \beta \to G\alpha$ in $\overrightarrow{\mathcal{D}}$ the category of direct $\mathcal{C}$-factorisations, $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$ is either empty or connected. Similarly G is a **fibring Reedy functor** if every $\alpha \in \mathrm{obj}(\mathcal{C})$, for every $\beta \in \mathrm{obj}(\mathcal{D})$, and for every map $\sigma : G\alpha \to \beta$ in $\overleftarrow{\mathcal{D}}$ the category of inverse $\mathcal{C}$-factorisations, $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, \sigma)$ is either empty or connected.

Definition 6.12. Let $G : \mathcal{C} \to \mathcal{D}$ be a cofibring Reedy functor, let $\alpha \in \mathrm{obj}(\mathcal{C})$ with $\deg(G\alpha) = k \in \mathbb{N}$. Suppose that G sends non-identity morphisms $\gamma \to \alpha$ to non-identity morphisms. The n**th latching subcategory**, $\mathcal{A}_n$, for $-1 \leq n \leq k-1$ is the full subcategory of the latching category of $G\alpha$ containing the both the objects with source of degree at most n and objects which are in the image of the induced functor G_*.

Let $-1 \leq n < k-1$, and let $\sigma : \beta \to G\alpha$ be an object in $\mathcal{A}_n$ not contained in $\mathcal{A}_{n-1}$. As G is a cofibring Reedy functor the category of direct $\mathcal{C}$-factorisations of (α, σ) is either connected or empty. Let S_{n+1} be the set of such σ for which it is connected and T_{n+1} the set of such σ for which it is empty.

The $(n+1)$**th intermediate latching subcategory**, $\mathcal{B}_{n+1}$, for $-1 \leq n < k-1$ is the full subcategory of the latching category of $G\alpha$ containing both the objects of $\mathcal{A}_n$ and the elements of S_{n+1}.

Remark 6.13. Note that $\mathcal{A}_{-1}$ contains only objects in the image of G_* but may contain morphisms between them not in the image. We also have that $\mathcal{A}_{k-1} = \partial(\overrightarrow{D} \downarrow G\alpha)$.
The above definition gives a nested sequence of subcategories of the latching category of $G\alpha$,

$$\mathcal{A}_{-1} \subseteq \mathcal{B}_0 \subseteq \mathcal{A}_0 \subseteq \ldots \subseteq \mathcal{B}_{k-2} \subseteq \mathcal{A}_{k-2} \subseteq \mathcal{B}_{k-1} \subseteq \mathcal{A}_{k-1} = \partial(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$$

The functor G_* clearly factors through $\mathcal{A}_{-1}$, we will abuse notation by writing $G_* : \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \to \mathcal{A}_{-1}$.

Proposition 6.14. *Let $G : \mathcal{C} \to \mathcal{D}$ be a cofibring Reedy functor, let $\mathcal{M}$ be a model category, let $Z \in \mathrm{obj}(\mathcal{M}^{\mathcal{D}})$, and let $\alpha \in \mathrm{obj}(\mathcal{C})$ such that G takes non-identity morphisms $\gamma \to \alpha$ to non-identity morphisms. Then the functor $G_* : \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \to \mathcal{A}_{-1}$ is right cofinal (see Definition 1.27) so by Proposition 1.28 induces an isomorphism*

$$\operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)} G^* Z \cong \operatorname*{colim}_{\mathcal{A}_{-1}} Z.$$

Proof. Every object in $\mathcal{A}_{-1}$ can be written $G\sigma : G\beta \to G\alpha$ for some $\sigma : \beta \to \alpha$ in the latching category of α. By Proposition 6.9 $(\sigma \downarrow G_*) = \mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$. As G is a cofibring Reedy category $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$ is either empty or connected. As σ is a non-identity morphism in $\overrightarrow{\mathcal{C}}$ the pair $(\sigma : \beta \to \alpha, 1_{G\beta})$ is in $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$ so it is not empty. Hence G_* is right cofinal. □

Remark 6.15. The sequence of subcategory inclusions induces a sequence of natural morphisms

$$\operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)} G^* Z \cong \operatorname*{colim}_{\mathcal{A}_{-1}} Z \to \operatorname*{colim}_{\mathcal{B}_0} Z \to \operatorname*{colim}_{\mathcal{A}_0} Z \to \ldots$$
$$\ldots \to \operatorname*{colim}_{\mathcal{B}_{k-1}} Z \to \operatorname*{colim}_{\mathcal{A}_{k-1}} Z = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow G\alpha)} Z.$$

There is a minor error in the proof of the following lemma in [4] which we have rectified here.

Lemma 6.16. *Under the assumptions of Proposition 6.14 the morphisms in the sequence above of the form $\operatorname*{colim}_{\mathcal{A}_n} Z \to \operatorname*{colim}_{\mathcal{B}_{n+1}} Z$ are isomorphisms.*

Proof. Let $\sigma : \beta \to G\alpha$ be an object of $\mathcal{B}_{n+1}$. We will write $(\sigma \downarrow \mathcal{A}_n)$ for the comma category where $\mathcal{A}_n$ represents the inclusion functor $\mathcal{A}_n \hookrightarrow \mathcal{B}_{n+1}$. If σ is an object of $\mathcal{A}_n$ then $(\sigma, 1_\beta)$ is an initial object in $(\sigma \downarrow \mathcal{A}_n)$ and so $(\sigma \downarrow \mathcal{A}_n)$ is connected. Otherwise $\sigma \in S_{n+1}$, in which case the objects of the category $(\sigma \downarrow \mathcal{A}_n)$ are pairs (ν', μ) where $\nu' : \gamma' \to G\alpha$ is an object of $\mathcal{A}_n$ and $\mu : \sigma \to \nu'$ is a morphism in $\mathcal{B}_{n+1}$, that is μ is a morphism $\beta \to \gamma'$ in $\overrightarrow{\mathcal{D}}$ such that the following diagram commutes.

$$\begin{tikzcd} \beta \arrow[rr, "\mu"] \arrow[dr, "\sigma"'] & & \gamma' \arrow[dl, "\nu'"] \\ & G\alpha & \end{tikzcd}$$

The morphism μ must be a non-identity morphism otherwise $\sigma = \nu'$ but this cannot be the case as σ is not an object of $\mathcal{A}_n$. Hence $\deg(\gamma') > \deg(\beta) = n + 1$. By Definition 6.12 ν' must be in the image of G_* that is there exists an object $\nu : \gamma \to \alpha$ in $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ such that $G\nu = \nu'$. The proof in [4] claims that this shows that $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma) = (\sigma \downarrow \mathcal{A}_n)$, however ν may not be unique. It is sufficent that we have shown that there is a surjective functor $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma) \to (\sigma \downarrow \mathcal{A}_n)$. As G is a cofibring Reedy functor $(\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma))$ is either empty or connected. It is not empty as $(\sigma, 1_\beta) \in \mathrm{obj}(\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma))$. So $(\sigma \downarrow \mathcal{A}_i)$ is connected. Hence the inclusion functor $\mathcal{A}_n$ is right cofinal and by Proposition 1.28 the induced morphism $\operatorname{colim}_{\mathcal{A}_n} Z \to \operatorname{colim}_{\mathcal{B}_{n+1}} Z$ is an isomorphism. □

Lemma 6.17. *Under the assumptions of Proposition 6.14 there is the following pushout square.*

$$\begin{tikzcd} \coprod\limits_{(\beta \to G\alpha) \in T_{n+1}} \operatorname{colim}\limits_{\partial(\overrightarrow{\mathcal{D}} \downarrow \beta)} Z \arrow[r] \arrow[d] & \coprod\limits_{(\beta \to G\alpha) \in T_{n+1}} Z_\beta \arrow[d] \\ \operatorname{colim}\limits_{\mathcal{B}_{n+1}} Z \arrow[r] & \operatorname{colim}\limits_{\mathcal{A}_{n+1}} Z \end{tikzcd}$$

Proof. Let the top morphism be coproduct the latching morphisms of Z at β. As the objects of the latching category of β are non-identity morphisms of the form $\gamma \to \beta$ we have $\deg(\gamma) < \deg(\beta) = n + 1$. This gives us a functor $\partial(\overrightarrow{\mathcal{D}} \downarrow \beta) \to \mathcal{B}_{n+1}, (\gamma \to \beta) \mapsto (\gamma \to \beta \to G\alpha)$ which induces the left morphism in the above diagram. The inclusion functor $\mathcal{B}_{n+1} \hookrightarrow \mathcal{A}_{n+1}$ induces the bottom morphism. The elements of T_{n+1} are objects in $\mathcal{A}_{n+1}$ so there is a natural morphism on the right side of the square.

The objects of $\mathcal{A}_{n+1}$ are the elements of T_{n+1} and the objects of $\mathcal{B}_{n+1}$ so a morphism $\operatorname*{colim}_{\mathcal{A}_{n+1}} \to W$ in $\mathcal{M}$ is determined by its precompositions with the right and bottom morphisms in the diagram below. As the morphisms in $\mathcal{A}_{n+1}$ whose domains are in T_{n+1} are all identity morphisms and the only non-identity morphisms with codomain an element $(\beta \to G\alpha) \in T_{n+1}$ are the objects of the matching category $\partial(\overrightarrow{\mathcal{D}} \downarrow \beta)$, morphisms $\operatorname*{colim}_{\mathcal{B}_{n+1}} Z \to W$ and $\coprod_{(\beta\to G\alpha)\in T_{n+1}} Z_\beta \to W$ determine a morphism $\operatorname*{colim}_{\mathcal{A}_{n+1}} \to W$ if and only if their compositions from $\coprod_{(\beta\to G\alpha)\in T_{n+1}} \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow\beta)} Z$ agree. $\square$

We will classify the Reedy functors for which the induced functor of diagram categories over $\mathcal{M}$ is a left Quillen functor for every model category $\mathcal{M}$. We will show this by dualising the classification theorem from Hirschhorn and Volić.

Theorem 6.18. *If $G : \mathcal{C} \to \mathcal{D}$ is a cofibring Reedy functor, then the induced functor $G^* : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a left Quillen functor for every model category $\mathcal{M}$.*

Proof. As a model category $\mathcal{M}$ is complete so G^* has a right Kan extension along $1_{\mathcal{M}^{\mathcal{D}}}$. So by Proposition 1.25 G^* has a right adjoint.

The induced functor G^* preserves weak equivalences; to see this suppose $f : X \to Y$ is a weak equivalence in $\mathcal{M}^{\mathcal{D}}$. Then $G^*f : XG \to YG$ has components $(G^*f)_\alpha = f_{G\alpha} : X(G\alpha) \to Y(G\alpha)$ which are weak equivalences in $\mathcal{M}$ hence G^* preserves weak equivalences.

It remains to show that G^* preserves cofibrations.

Let $f : X \to Y$ be a cofibration in $\mathcal{M}^{\mathcal{D}}$. That is, for all objects β in $\mathcal{D}$ the relative latching map $X_\beta \amalg_{L^{\mathcal{D}}_\beta X} L^{\mathcal{D}}_\beta Y \to Y_\beta$ is a cofibration in $\mathcal{M}$.

So we want to show that $G^*f : XG \to YG$ is a cofibration in $\mathcal{M}^{\mathcal{C}}$. That is, for all objects α in $\mathcal{C}$ we want to show that the relative latching map $(G^*X)_\alpha \amalg_{L^{\mathcal{C}}_\alpha(G^*X)} L^{\mathcal{C}}_\alpha(G^*Y) \to (G^*Y)_\alpha$ is a cofibration. We will write $\mathcal{P}^{\mathcal{C}}_\alpha = (G^*X)_\alpha \amalg_{L^{\mathcal{C}}_\alpha(G^*X)} L^{\mathcal{C}}_\alpha(G^*Y)$.

Let $\alpha \in \operatorname{obj}(\mathcal{C})$ with $\deg(\alpha) = k \in \mathbb{N}$. We consider two cases:

(i) there exist non-identity morphisms $\nu : \gamma \to \alpha$ in $\overrightarrow{\mathcal{C}}$ such that $G\nu = 1_{G\alpha}$

(ii) G takes all non-identity morphisms $\nu : \gamma \to \alpha$ in $\overrightarrow{\mathcal{C}}$ to non-identity morphisms.

First we consider the case where there exist non-identity morphisms $\nu : \gamma \to \alpha$ in $\overrightarrow{\mathcal{C}}$ such that $G\nu = 1_{G\alpha}$. We call the set of such morphisms the G**-kernel**.

If $\nu : \gamma \to \alpha$ is in the G-kernel, then the objects of $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, G\nu)$ are precisely pairs of the form $(\mu, 1_{G\alpha})$ where $\mu : \kappa \to \alpha$ is an element of the G-kernel. To see this suppose that $\mu : \kappa \to \alpha$ is in the G-kernel, then the following diagram commutes

$$\begin{tikzcd} G\alpha \arrow[rr, equal, "G\mu = 1_{G\alpha}"] \arrow[dr, equal, "G\nu = 1_{G\alpha}"'] & & G\alpha \arrow[dl, equal, "1_{G\alpha}"] \\ & G\alpha & \end{tikzcd}$$

so $(\mu, 1_{G\alpha})$ is an object of $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, G\nu)$. Conversely, suppose that (u, v) is an object in $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, G\nu)$, then u is a non-identity morphism in $\overrightarrow{\mathcal{C}}$ and v is a morphism of $\overrightarrow{\mathcal{D}}$ such that the following diagram commutes.

$$\begin{tikzcd} G\alpha \arrow[rr, "Gu"] \arrow[dr, equal, "G\nu = 1_{G\alpha}"'] & & \xi \arrow[dl, "v"] \\ & G\alpha & \end{tikzcd}$$

As G is a Reedy functor Gu is a morphism in $\overrightarrow{\mathcal{D}}$ so $\deg(\xi) \geq \deg(G\alpha)$. As v is in $\overleftarrow{\mathcal{D}}$ we have $\deg(\xi) \leq \deg(G\alpha)$. Hence $\deg(\xi) = \deg(G\alpha)$. As the only morphisms in $\overleftarrow{\mathcal{D}}$ which do not raise degree are identity morphisms $v = 1_{G\alpha}$ we have $Gu = 1_{G\alpha}(Gu) = 1_{G\alpha}$. Hence u is in the G-kernel. As the latching object is a colimit it corresponds to a cocone $\{\chi_\nu : X_{G\gamma} \to L^{\mathcal{C}}_\alpha(G^*X)\}_{(\nu:\gamma\to\alpha)\in \mathrm{obj}(\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha))}$. Let $\nu : \gamma \to \alpha$ and $\mu : \kappa \to \alpha$ be elements of the G-kernel such that there is a morphism $\tau : \nu \to \mu$ in $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$, that is a morphism $\tau : \gamma \to \kappa$ in $\overrightarrow{\mathcal{C}}$. Clearly $G\tau = 1_{G\alpha}$ so we have the commutative diagram.

$$\begin{tikzcd} X_{G\alpha} \arrow[rr, equal] \arrow[dr, "\chi_\nu"'] & & X_{G\alpha} \arrow[dl, "\chi_\mu"] \\ & L^{\mathcal{C}}_\alpha(G^*X) & \end{tikzcd}$$

Hence $\chi_\nu = \chi_\mu$. As G is a cofibering Reedy functor $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, G\nu)$ is connected. That is, any there is a zig-zag of morphisms $\overrightarrow{\mathcal{C}}$ between the sources of any two morphisms in the G-kernel. Hence $\chi_\nu = \chi_\mu$ for any two elements ν and μ of the G-kernel.

Let $0 \leq n < k$. Let $\rho : \omega \to \alpha$ be an object of $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ with $\deg(G\omega) = n$. Suppose that for all $\mu : \kappa \to \alpha$ in $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ such that $\deg(G\kappa) > n$, there exists

$\nu : \gamma \to \alpha$ in the G-kernel and $\tau : \kappa \to \gamma$ in $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ such that the following diagram commutes.

$$\begin{tikzcd} X_{G\kappa} \arrow[r, "XG\tau"] \arrow[dr, "\chi_\mu"'] & X_{G\alpha} \arrow[d, "\chi_\nu"] \\ & L^{\mathcal{C}}_{\alpha}(G^*X) \end{tikzcd}$$

As $\chi_\nu = \chi_\mu$ for any ν and μ in the G-kernel, $\chi_\nu(XG\tau) = \chi_\mu = \chi_{\nu'}(XG\tau') = \chi_\nu(XG\tau')$ for any other choice of $\nu' : \gamma' \to \alpha$ in the G-kernel and $\tau' : \kappa \to \gamma'$.

Consider the catgeory of direct $\mathcal{C}$ factorisations of $G\rho$. Clearly $(\rho, G\nu)$ for any ν in the G-kernel and $(1_\omega, G\rho)$ are objects of $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, G\rho)$. As G is a cofibring Reedy functor $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, G\rho)$ is connected so there is a zig-zag between $(1_\omega, G\rho)$ and $(\rho, G\nu)$ for $\nu : \gamma \to \alpha$ in the G-kernel.

If $(u : \xi \to \alpha, v : G\omega \to G\xi)$ and $(w : \psi \to \alpha, x : G\omega \to G\psi)$ are in $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, G\rho)$ and $\deg(G\xi) > \deg(\psi)$ then a morphism $(u, v) \to (w, x)$ is a morphism $\tau : \xi \to \phi$ in $\overrightarrow{\mathcal{C}}$ so as a Reedy functor $G\tau : G\xi \to X\psi$ cannot lower degree, a contradiction. Hence there is no such morphism. Thus a zig-zag between $(1_\omega, G\rho)$ and $(\rho, G\nu)$ for $\nu : \gamma \to \alpha$ in the G-kernel can be split into zig-zags

$$\begin{tikzcd} & (u_1, v_1) \arrow[dl] \arrow[dr] & & \arrow[dl] \cdots \arrow[dr] & & (u_q, v_q) \arrow[dl] \arrow[dr, "\eta"] & \\ (1_\omega, G\rho) & & (u_2, v_2) & & (u_{q-1}, v_{q-1}) & & (\mu, \phi) \end{tikzcd}$$

and

$$\begin{tikzcd} & (w_1, x_1) \arrow[dl] \arrow[dr] & & \arrow[dl] \cdots \arrow[dr] & & (w_r, x_r) \arrow[dl] \arrow[dr] & \\ (\mu, \phi) & & (w_2, x_2) & & (w_{r-1}, x_{r-1}) & & (\rho, G\nu) \end{tikzcd}$$

where $\mu : \kappa \to \alpha$ with $\deg(G\kappa) > n$, and for all $u_i : \xi_i \to \alpha$ we have $\deg(G\xi_i) = n$.

If (u, v) is an object of $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, G\rho)$ where $u : \xi \to \alpha$ and $\deg(G\omega) = \deg(G\xi)$ then as $v : G\xi \to G\omega$ is degree preserving it must be the identity morphism on $G\omega$ so $G\rho = Gu$. Hence $\chi_\rho = \chi_{u_1} = \ldots = \chi_{u_r}$ by the same argument used to show that the components of the cocone corresponding to the elements of the G-kernel are equal. Hence, by assumption, there exists $\nu' : \gamma' \to \alpha$ in the G-kernel and $\tau : \kappa \to \gamma'$ in $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ such that the following diagram commutes.

$$\begin{tikzcd}
X_{G\omega} \arrow[r, "XG\eta"] \arrow[drr, bend right, "\chi_\rho"'] & X_{G\kappa} \arrow[r, "XG\tau"] \arrow[dr, "\chi_\mu"'] & X_{G\alpha} \arrow[d, "\chi_{\nu'}"] \\
 & & L_\alpha^{\mathcal{C}}(G^*X)
\end{tikzcd}$$

Similarly to before, the independence of χ_ν from the choice of ν in the G-kernel tells us that $\chi_\nu(XG\tau)(XG\eta) = \chi_\rho = \chi_\nu(XG\tau')(XG\eta')$ for any such μ', τ', and η'. We have established that for all objects $\sigma : \beta \to \alpha$ in the latching category $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ and all elements $\nu : \gamma \to \alpha$ in the G-kernel, there exists $\nu' : \gamma' \to \alpha$ in the G-kernel and $\tau : \beta \to \gamma'$ in $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ such that the following diagram commutes.

$$\begin{tikzcd}
X_{G\beta} \arrow[r, "XG\tau"] \arrow[dr, "\chi_\mu"'] & X_{G\alpha} \arrow[d, "\chi_\nu"] \\
 & L_\alpha^{\mathcal{C}}(G^*X)
\end{tikzcd}$$

Fix some ν in the G-kernel. Let $h : L_\alpha^{\mathcal{C}}(G^*X) \to W$ for $W \in \mathrm{obj}(\mathcal{M})$, this map corresponds to a cocone $\{h_\sigma : G\beta \to L_\alpha^{\mathcal{C}}(G^*X)\}_{(\sigma:\beta\to\alpha)\in\mathrm{obj}(\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha))}$. We will call the precomposition of h with the injection map $\hat{h} : X_{G\alpha} \xrightarrow{\chi_\nu} L_\alpha^{\mathcal{C}}(G^*X) \xrightarrow{h} W$. Precomposing this with the latching map at α we have another morphism $L_\alpha^{\mathcal{C}}(G^*X) \xrightarrow{\varphi} X_{G\alpha} \xrightarrow{\hat{h}} W$ which also corresponds to a cocone $\{\hat{h}_\sigma : G\beta \to L_\alpha^{\mathcal{C}}(G^*X)\}_{(\sigma:\beta\to\alpha)\in\mathrm{obj}(\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha))}$. By the commutative properties of cocones for each σ in the latching category at α, using the τ gving us the above commutative diagram, and noting that $\varphi\chi_\nu = 1_{X_{G\alpha}}$ we have $\hat{h}_\sigma = \hat{h}\varphi\chi_\sigma = h\chi_\nu\varphi\chi_\sigma = h\chi_\nu\varphi\chi_\nu(GX\tau) = h\chi_\nu(GX\tau) = h\chi_\sigma = h_\sigma$. As the cocones are the same, they correspond to the same morphism so $\hat{h}\varphi = h$. Consider the morphism of sets $- \circ \varphi : \mathcal{M}(X_{G\alpha}, W) \to \mathcal{M}(L_\alpha^{\mathcal{C}}(G^*X), W), g \mapsto g\varphi$. Given $h \in \mathcal{M}(L_\alpha^{\mathcal{C}}(G^*X), W)$ we can form $\hat{h}$ as above, then $\hat{h}\varphi = h$ so $- \circ \varphi$ is surjective. Suppose that $g, g' \in \mathcal{M}(X_{G\alpha}, W)$ such that $g\varphi = g'\varphi$, then for ν in the G-kernel $g = g\varphi\chi_\nu = g'\varphi\chi_{nu} = g'$. So $-\circ\varphi$ is injective, so $-\circ\varphi$ is an isomorphism of sets. Hence there is an isomorphism $X_{G\alpha} \cong L_\alpha^{\mathcal{C}}(G^*X)$ in $\mathcal{M}$.

Thus we have $\mathcal{P}_\alpha^{\mathcal{C}} = G^*X_\alpha \amalg_{L_\alpha^{\mathcal{C}}G^*X} L_\alpha^{\mathcal{C}}G^*Y = X_{G\alpha} \amalg_{X_{G\alpha}} Y_{G\alpha}$. Hence the relative latching map at $\mathcal{P}_\alpha^{\mathcal{C}} \to Y_{G\alpha}$ is an isomorphism, and in particular this means it is a cofibration.

Now we consider the case where all non identity morphisms $\nu : \gamma \to \alpha$ in $\overrightarrow{\mathcal{C}}$ we have $G\nu \neq 1_{G\alpha}$. In this case the G-kernel is empty and so the downward induction used in the previous section cannot be used.

The latching objects of G^*X and G^*Y respectively at α are

$$L^{\mathcal{C}}_{\alpha}G^*X = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)} G^*X \quad \text{and} \quad L^{\mathcal{C}}_{\alpha}G^*Y = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)} G^*Y.$$

Similarly, the latching objects of X and Y respectively at $G\alpha$ are

$$L^{\mathcal{D}}_{G\alpha}X = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow G\alpha)} X \quad \text{and} \quad L^{\mathcal{D}}_{G\alpha}Y = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow G\alpha)} Y.$$

The functor $G_* : \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \to \partial(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$ (see Remarks 6.8 and 6.10) induces maps

$$L^{\mathcal{C}}_{\alpha}G^*X = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)} G^*X \to \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow G\alpha)} X = L^{\mathcal{D}}_{G\alpha}X$$
$$L^{\mathcal{C}}_{\alpha}G^*Y = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)} G^*Y \to \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow G\alpha)} Y = L^{\mathcal{D}}_{G\alpha}Y.$$

We have the following commutative diagram.

$$\begin{tikzcd}
 & L^{\mathcal{C}}_{\alpha}G^*X \arrow[rr] \arrow[dl] \arrow[dd] & & (G^*X)_{\alpha} \arrow[dl, equal] \arrow[dd] & \\
L^{\mathcal{D}}_{G\alpha}X \arrow[rr, crossing over] \arrow[dd] & & X_{G\alpha} & & \\
 & L^{\mathcal{C}}_{\alpha}G^*Y \arrow[rr] \arrow[dl] & & \mathcal{P}^{\mathcal{C}}_{\alpha} \arrow[dl] \arrow[dr] & \\
L^{\mathcal{D}}_{G\alpha}Y \arrow[rr] & & \mathcal{P}^{\mathcal{D}}_{G\alpha} \arrow[from=uu, crossing over] \arrow[dr] & & (G^*Y)_{\alpha} \arrow[dl, equal] \\
 & & & Y_{G\alpha} &
\end{tikzcd}.$$

As $f : X \to Y$ is a cofibration in $\mathcal{M}^{\mathcal{D}}$ we know that the morphism $\mathcal{P}^{\mathcal{D}}_{G\alpha} \to Y_{G\alpha}$ is a cofibration in $\mathcal{M}$. So it is sufficient to show that $\mathcal{P}^{\mathcal{C}}_{\alpha} \to \mathcal{P}^{\mathcal{D}}_{G\alpha}$ is a cofibration. Let $\mathcal{P}_{\mathcal{A}_k}$ and $\mathcal{P}_{\mathcal{B}_k}$ be the pushouts

$$\begin{tikzcd}
\operatorname*{colim}_{\mathcal{A}_k} X \arrow[r] \arrow[d] & X_{G\alpha} \arrow[d] \\
\operatorname*{colim}_{\mathcal{A}_k} Y \arrow[r] & \mathcal{P}_{\mathcal{A}_k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname*{colim}_{\mathcal{B}_k} X \arrow[r] \arrow[d] & X_{G\alpha} \arrow[d] \\
\operatorname*{colim}_{\mathcal{B}_k} Y \arrow[r] & \mathcal{P}_{\mathcal{B}_k}
\end{tikzcd}$$

respectively and consider the sequence of maps from 6.15 for X and Y. This gives a factorisation of $\mathcal{P}^{\mathcal{C}}_{\alpha} \to \mathcal{P}^{\mathcal{D}}_{G\alpha}$ as $\mathcal{P}^{\mathcal{C}}_{\alpha} \to \mathcal{P}_{\mathcal{A}_{-1}} \to \mathcal{P}_{\mathcal{B}_0} \to \mathcal{P}_{\mathcal{A}_0} \to \ldots \to \mathcal{P}_{\mathcal{A}_{k-2}} \to \mathcal{P}_{\mathcal{B}_{k-1}} \to \mathcal{P}^{\mathcal{D}}_{G\alpha}$.

By Lemma 6.16, the morphisms $\operatorname{colim}_{\mathcal{A}_n} X \to \operatorname{colim}_{\mathcal{B}_{n+1}} X$ and $\operatorname{colim}_{\mathcal{A}_n} Y \to \operatorname{colim}_{\mathcal{B}_{n+1}} Y$ are isomorphisms for all $-1 \leq n < k-1$ and hence the morphisms $\mathcal{P}_{\mathcal{A}_n} \to \mathcal{P}_{\mathcal{B}_{n+1}}$ are isomorphisms for all $-1 \leq n < k-1$. In particular this means that these morphisms are cofibrations.

Let Q and R be the following pushouts.

$$\begin{tikzcd}
\operatorname*{colim}_{\mathcal{B}_{n+1}} X \arrow[r] \arrow[d] & \operatorname*{colim}_{\mathcal{A}_{n+1}} X \arrow[d] \\
\operatorname*{colim}_{\mathcal{B}_{n+1}} Y \arrow[r] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod_{(\sigma:\beta\to G\alpha)\in T_{n+1}} \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow\beta)} X \arrow[r] \arrow[d] & \coprod_{(\sigma:\beta\to G\alpha)\in T_{n+1}} X_\beta \arrow[d] \\
\coprod_{(\sigma:\beta\to G\alpha)\in T_{n+1}} \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow\beta)} Y \arrow[r] & R
\end{tikzcd}$$

Consider the following commutative diagram in $\mathcal{M}$.

$$\begin{tikzcd}
\coprod_{(\sigma:\beta\to G\alpha)\in T_{n+1}} \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow\beta)} X \arrow[rrr, "t"] \arrow[ddr, "\psi"] \arrow[ddd, "v"] & & & \coprod_{(\sigma:\beta\to G\alpha)\in T_{n+1}} X_\beta \arrow[dl, "c"] \arrow[ddd, "u"] \arrow[ddr, "\varphi"] & \\
& & R \arrow[drr, "a"] \arrow[ddd, "h"] & & \\
& \coprod_{(\sigma:\beta\to G\alpha)\in T_{n+1}} \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{D}}\downarrow\beta)} Y \arrow[ur, "d"] \arrow[rrr, "t'"] \arrow[ddd, "v'"] & & & \coprod_{(\sigma:\beta\to G\alpha)\in T_{n+1}} Y_\beta \arrow[ddd, "u'"] \\
\operatorname*{colim}_{\mathcal{B}_{n+1}} X \arrow[rrr, "s"] \arrow[ddr, "\lambda"] & & & \operatorname*{colim}_{\mathcal{A}_{n+1}} X \arrow[dl, "e"] \arrow[ddr, "p"] & \\
& & Q \arrow[drr, "b"] & & \\
& \operatorname*{colim}_{\mathcal{B}_{n+1}} Y \arrow[ur, "g"] \arrow[rrr, "s'"] & & & \operatorname*{colim}_{\mathcal{A}_{n+1}} Y
\end{tikzcd}$$

By Lemma 6.17 the front and back squares are pushouts. The square

$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "h"] & \coprod\limits_{(\sigma:\beta\to G\alpha)\in T_{n+1}} X_\beta \arrow[d, "u'"] \\ Q \arrow[r, "b"] & \operatorname*{colim}\limits_{\mathcal{A}_{n+1}} Y \end{tikzcd}$$

is also a pushout. We show this by showing that the universal property of pushouts (see Example 1.8 (iii)) is satisfied. Suppose that W is an object in $\mathcal{M}$, and $m : Q \to W$ and $n : \coprod\limits_{(\sigma:\beta\to G\alpha)\in T_{n+1}} X_\beta \to W$ are morphisms in $\mathcal{M}$ such that $mh = na$. We have $(mg)v' = mhd = nad = nt'$ so as the front square is a pushout, mg and n induce a morphism $\phi : \operatorname*{colim}\limits_{\mathcal{A}_{n+1}} \to W$ such that $\phi s' = mg$ and $\phi u' = n$.

We have $\phi bes = \phi ps = \phi s'\lambda = mg\lambda = mes$ and $\phi beu = \phi pu = \phi u'\varphi = \phi ua'c = nac = mhc = meu$. As the back square is a pushout this tells us that $\phi be = me$. Further since $\phi bg = \phi s' = mg$ and Q is a pushout we have $\phi b = m$.

Suppose that $\omega : \operatorname*{colim}\limits_{\mathcal{A}_{n+1}} Y \to W$ is a morphism in $\mathcal{M}$ such that $\omega u' = n$ and $\omega b = m$. Then $\omega u' = n = \phi u'$ and $\omega s' = \omega bg = mg = \phi s'$. As the front square is a pushout $\omega = \phi$. Hence the required morphism exists and is unique satisfying the universal property.

Note that the morphism $a : R \to \coprod\limits_{(\sigma:\beta\to G\alpha)\in T_{n+1}} X_\beta$ is a coproduct of relative latching maps and so is itself a relative latching map. So a is a cofibration in $\mathcal{M}$ as f is a cofibration in $\mathcal{M}^{\mathcal{D}}$. As the pushout of a cofibration is a cofibration, the morphism $b : Q = \operatorname*{colim}\limits_{\mathcal{A}_{n+1}} X \amalg_{\left(\operatorname*{colim}\limits_{\mathcal{B}_{n+1}} X\right)} \operatorname*{colim}\limits_{\mathcal{B}_{n+1}} Y \to \operatorname*{colim}\limits_{\mathcal{A}_{n+1}} Y$ is also a cofibration.

For $-1 \leq n < k-1$ the morphism $\mathcal{P}_{\mathcal{B}_{n+1}} \to \mathcal{P}_{\mathcal{A}_{n+1}}$ is defined by the following commutative diagram.

$$\begin{tikzcd}
\operatorname*{colim}\limits_{\mathcal{B}_{n+1}} X \arrow[rr] \arrow[dr] \arrow[dd] & & X_{G\alpha} \arrow[dr, equal] \arrow[dd] & \\
& \operatorname*{colim}\limits_{\mathcal{A}_{n+1}} X \arrow[rr] \arrow[dd] & & X_{G\alpha} \arrow[dd] \\
\operatorname*{colim}\limits_{\mathcal{B}_{n+1}} Y \arrow[rr] \arrow[dr] & & \mathcal{P}_{\mathcal{B}_{n+1}} \arrow[dr] & \\
& \operatorname*{colim}\limits_{\mathcal{A}_{n+1}} Y \arrow[rr] & & \mathcal{P}_{\mathcal{A}_{n+1}}
\end{tikzcd}$$

From above we have that $\operatorname*{colim}\limits_{\mathcal{A}_{n+1}} X \amalg_{\left(\operatorname*{colim}\limits_{\mathcal{B}_{n+1}} X\right)} \operatorname*{colim}\limits_{\mathcal{B}_{n+1}} Y \to \operatorname*{colim}\limits_{\mathcal{A}_{n+1}} Y$ is a cofibration. As an isomorphism, the identity $1_{X_{G\alpha}}$ is a cofibration. As the front and back squares are pushout squares by Lemma 4.27, the morphism $\mathcal{P}_{\mathcal{B}_{n+1}} \to \mathcal{P}_{\mathcal{A}_{n+1}}$ is a cofibration.

As a composition of cofibrations the morphism $\mathcal{P}^{\mathcal{C}}_{\alpha} \to \mathcal{P}^{\mathcal{D}}_{G\alpha}$ is a cofibration. Hence the relative latching map at α is a cofibration.

So the relative latching map $\mathcal{P}^{\mathcal{C}}_{\alpha} \to Y_{G\alpha}$ is a cofibration for all objects α in $\mathcal{C}$. Hence G^* preserves cofibrations and so is a left Quillen functor. $\square$

Bibliography

[1] W. G. Dwyer and J. Spalinski. Homotopy theories and model categories. *Handbook of algebraic topology*, 73:126, 1995.

[2] A. Hatcher. *Algebraic topology.* Cambridge University Press, 2002.

[3] P. S. Hirschhorn. *Model categories and their localizations.* Number 99 in Mathematical Surveys and Monographs. American Mathematical Society, 2003.

[4] P. S. Hirschhorn and I. Volić. Functors between Reedy model categories of diagrams. *ArXiv e-prints*, Nov. 2015.

[5] M. Hovey. *Model categories.* Number 63. American Mathematical Society, 1999.

[6] L. G. Lewis. Is there a convenient category of spectra? *Journal of pure and applied algebra*, 73(3):233–246, 1991.

[7] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proceedings of the London Mathematical Society*, 82(2):441–512, 2001.

[8] D. G. Quillen. *Homotopical algebra*, volume 43 of *Lecture Notes in Mathematics*. Springer, 1967.

[9] M. Saunders. *Categories for the working mathematician.* Springer, 1971.

[10] E. H. Spanier. *Algebraic topology.* McGraw-Hill, 1966.